THE UNIVERSITY OF ALBERTA

A STUDY OF PION-PION SCATTERING INVOLVED IN K-MESON DECAYS

by

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A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS
EDMONTON, ALBERTA
SEPTEMBER, 1968

ABSTRACT

Certain K-meson decay processes are investigated under the assumption that the enhancement of the basic first-order process is a result of pion-pion interactions. The s-wave π-π scattering process is parametrized by means of plausible K-π-π form factors, the parameters are adjusted to predict the experimentally observed $K_1^0 - K_2^0$ mass difference, and the corresponding s-wave phase shifts are calculated. The form factors used are found to tend to lead to small positive π-π scattering lengths and small values of the s-wave phase shift at a π-π energy equal to the kaon mass. In $K^+ \rightarrow 3\pi$ decay, π-π scattering is parametrized by means of the I=0 and I=2 scattering lengths, and if the pions are assumed to interact only in s-waves, a comparison with experiment shows that $a_2 - a_0 \simeq 0.7$. However, if p-wave scattering is also included, a K-ρ-π coupling constant of the order of the basic K-3π coupling constant can yield widely different values for $a_2 - a_0$. A simple model for π-π scattering in $K_{\ell 4}$ decay leads to a divergent s-wave form factor but a well-behaved phase shift.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. A.N. Kamal, for suggesting this problem and for counselling and assisting me throughout the course of this work.

I am also indebted to the Computing Centre at the University of Alberta for the use of their facilities.

The financial assistance of the National Research Council of Canada is gratefully acknowledged.

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CHAPTER I INTRODUCTION

With the direct pion-pion scattering experiments not yet feasible, any study of the pion-pion interaction must be done by indirect methods. Fortunately, pions are involved in a great many processes for which some experimental data is available, so that one can analyze several processes from different points of view, and thus at least partially overcome the disadvantage of not having the direct scattering data available.

The weak decay of the K-meson into pions and, perhaps, some other particles has two properties which make it especially well suited to the study of pion-pion interactions. First of all, the kaon has spin 0 and isospin ½, which leads to uncomplicated final states, and, secondly, the kaon mass is less than four times the mass of the pion, so that the energy of the pions emitted in K decays, particularly in $K \to 3\pi$ decays, is small. This leads to a considerable simplification in the calculations.

Three processes involving a kaon and pions will be considered in this work. The first of these is the self-energy process of the neutral kaon, which leads to a

$K_1^0 - K_2^0$ mass difference. Next is the $K \to 3\pi$ decay, and finally, the $K \to 2\pi + \ell + \bar{\nu}$ (K_{ℓ_4}) decay. To learn something about pion-pion scattering, the point of view will be taken that the kaons first break up in a point weak interaction, and then the pions interact in the final state via strong interactions to produce the energy dependence of the process(1). Since there is as yet no successful theory of strong interactions, one must propose some plausible model for pion-pion scattering, involving some unknown parameters. Then the result of the analysis of the kaon decay involving the pion-pion final-state interactions can be compared with experiment, and the parameters can be fixed. If there are not too many parameters in the model for π-π scattering, some meaningful conclusions about the nature of the π-π interaction can be drawn.

In Chapter II, the $K_1^0 - K_2^0$ mass difference is considered in its relation to π-π scattering. It is shown that the neutral kaon self-energy processes are dominated by the two-pion virtual state in the K_1^0 self-energy. Then the mass difference, which is simply the difference between the K_1^0 and K_2^0 self-energies, is given in good approximation by the two-pion virtual state contribution to the K_1^0 self-energy. Several plausible parametrizations for the K-π-π

form factor are suggested and the $K_1^0 - K_2^0$ mass difference is calculated using these form factors. This result is compared with the experimentally determined mass difference and hence the parameters are determined. The form factor is shown to be related to the s-wave π-π phase shift by an integral equation, which is inverted to give the phase shifts explicitly.

In Chapter III, $K \to 3\pi$ decay is analyzed including pion-pion interactions in both the s-wave and p-wave. For the s-wave interactions, a parametrization involving a_0 and a_2, the I=0 and I=2 ($I \equiv$ isospin) scattering lengths, is used. The p-wave interaction calculations are done assuming that the ρ resonance dominates the interaction, and therefore the parameter used here is the K-ρ-π coupling constant. The matrix element is found with the effects of both the s and p-wave interactions included.

In Chapter IV, the K_{ℓ_4} decay is discussed, again including both s and p-wave final-state interactions. The s-wave π-π interaction is parametrized by the I=0 and I=2 scattering lengths but it is found that the resulting matrix element does not depend on a_2. The p-wave interaction is assumed to be dominated by the ρ resonance, and the parameter involved here is the K-ρ-ℓ-$\bar{\nu}$ coupling constant.

CHAPTER II $K_1^0 - K_2^0$ MASS DIFFERENCE

Within the framework of strong interaction theory, the masses of the K_1^0 and K_2^0 mesons are degenerate. This degeneracy is removed by self-energy processes of second and higher (even) order in the weak interaction. Of course, since the weak coupling constant is so small, processes of higher than second order are neglected when calculating the self-energy. The different decay behaviour of the K_1^0 and K_2^0 mesons leads to different self-energies, and hence to a mass difference. Contributions to the self-energy of the kaons can come from mesonic, leptonic, or baryonic virtual states. However, it turns out that the $K_1^0 - K_2^0$ mass difference can be considered, in good approximation, to arise only from one self-energy process - the virtual decay of the K_1^0 into two pions.

The mesonic self-energy processes involve virtual states of one or more pions. For this problem, CP noninvariance effects are small$^{(2)}$, so that one need only consider even numbers of pions for the K_1^0 self-energy processes and odd numbers of pions for the K_2^0 self-energy processes. The K_1^0 is known to decay almost exclusively into two pions, and therefore one anticipates that the two-pion process accounts for most of the K_1^0 self-energy. The main me-

sonic decay of the K_2^0 is the three-pion decay, but the rate for this process is very much less than the rate for $K_1^0 \rightarrow 2\pi$ so that the contribution to the mass difference from the three-pion process is expected to be negligible compared with the contribution from the two-pion process in the K_1^0 self-energy. The one-pion self-energy process, the so-called "pole" term, in the K_2^0 self-energy is not expected to make a significant contribution to the mass difference because the process $K^+ \rightarrow \pi^+ + e^+ + e^-$, which contains a $K - \pi$ vertex, goes at a very much slower rate than $K_1^0 \rightarrow 2\pi$ and leads to a mass difference at least one order of magnitude smaller than the observed value(3).

Weak interactions have never been observed to produce uncharged lepton pairs (i.e. no neutral weak leptonic currents are known to exist), so that one can make the reasonable assumption that the pure leptonic virtual states give no contribution to the kaon self-energy. If the $\Delta S = \Delta Q$ rule is exact, the semi-leptonic processes (i.e. $K^0 \rightarrow \pi + \ell + \nu$) do not contribute to the mass difference, since the self-energies arising from these processes are the same for both K_1^0 and K_2^0. This can be seen by noting that, since the K_1^0 and K_2^0 mesons are combinations of the K^0 and $\overline{K}^0$ mesons, the decay of either K_1^0 or K_2^0 into a particular π-ℓ-ν state involves the sum (K_1^0 processes) or the difference (K_2^0 processes) of a $\Delta S = \Delta Q$ amplitude and a

$\Delta S \neq \Delta Q$ amplitude. If the $\Delta S \neq \Delta Q$ amplitude is not zero, the self-energies are different; if the $\Delta S \neq \Delta Q$ amplitude is zero, the self-energies are the same. Even if the $\Delta S = \Delta Q$ rule is not exact, the leptonic decay rates are much slower than the $K_1^0 \rightarrow 2\pi$ decay rate, so that any leptonic self-energy effects may be neglected.

Baryon-antibaryon virtual states give no contribution to the self-energies because of the large energy denominators that they produce.

The above arguments show that the $K_1^0 - K_2^0$ mass difference may be calculated in excellent approximation simply by calculating the two-pion intermediate state contribution to the self-energy of the K_1^0 meson. Having obtained this result, one can now suggest some plausible parametrization for the pion-pion interaction, calculate an expression for the K_1^0 self-energy, and hence adjust the parameters to make this self-energy equal to the observed $K_1^0 - K_2^0$ mass difference. Experiments(4) have shown this mass difference to be given by

$$2\tau_1 \Delta M \;=\; 2\tau_1\{M(K_1^0) - M(K_2^0)\} \;=\; -0.96 \pm 0.04 \;,$$

where τ_1 is the lifetime of the K_1^0 meson. According to a technique outlined below, the quantity $2\tau_1 \Delta M$ can be calcu-

lated if the K-π-π form factor is known, and this form factor can be directly related to the pion-pion phase shift. The two virtual pions are produced in an $\ell=0$ angular momentum state since the kaon is spinless, so that by extended Bose statistics, they must be in a state of even isospin. Assuming that the $|\Delta I| = \frac{1}{2}$ rule is valid, the isospin of the two-pion state is 0, since the isospin of the kaon is ½. Therefore, information on I=0, s-wave pion-pion scattering can be obtained from a knowledge of the $K_1^0 - K_2^0$ mass difference.

The self-energy operator, $\Sigma(s)$, of the K_1^0 meson is assumed to obey an unsubtracted dispersion relation

$$\Sigma(s) = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\mathrm{Im}\Sigma(s')}{s' - s - i\varepsilon}\, ds' ,$$

where $s = 4(k^2 + m^2)$ is the square of the total centre-of-mass energy of the two pions. The lower limit of integration is the value of s at the threshold for the process $K_1^0 \rightarrow 2\pi$. Then the kaon self-energy is given by $\Sigma(s)$ evaluated at $s = M^2$ and, as was first shown by Barger and Kazes(5), this self-energy is given in terms of the K-π-π form factor by

$$\Sigma(M^2) = \alpha\int_0^{\infty} \frac{|F(k^2)|^2\, k^2\, dk}{(k^2 + m^2)^{\frac{1}{2}}(k^2 + m^2 - \frac{M^2}{4} - i\varepsilon)} . \quad \text{(II.1)}$$

Here, M is the unrenormalized kaon mass, m is the pion mass, and α is a normalization constant. $\Sigma(M^2)$ is related to the $K_1^0 - K_2^0$ mass difference by

$$2\tau_1 \Delta M = \frac{-\operatorname{Re}\Sigma(M^2)}{\operatorname{Im}\Sigma(M^2)} . \tag{II.2}$$

$\operatorname{Re}\Sigma(M^2)$ is the principal part of the integral (II.1), and

$$\operatorname{Im}\Sigma(M^2) = \tfrac{1}{2}\frac{|F(k^2)|^2 k}{(k^2 + m^2)^{\frac{1}{2}}} , \tag{II.3}$$

with $k^2 = \frac{M^2}{4} - m^2$.

With the assumption of elastic unitarity, the form factor is related to the s-wave pion-pion phase shift(6) by an Omnès equation(7) of the form

$$|F(k^2)| = \exp\left(\frac{2k^2}{\pi} P\int_0^\infty \frac{\delta(k')dk'}{k'(k'^2 - k^2)}\right) . \tag{II.4}$$

Notice that the phase shift is assumed to be real (elastic scattering). The phase shift is not determined completely by the form factor but is arbitrary within some function of k which will cause the principal part of the above integral to vanish.

At this stage one could use some plausible parametrization for the pion-pion scattering phase shift (Barger and Kazes(5)), or solve the π-π scattering problem assuming a dynamical model (Rockmore and Yao(8); Kang and Land(9)),

and thereby, in effect, determine the phase shift. Then the form factor could be evaluated using the above Omnès equation and hence, the $K_1^0 - K_2^0$ mass difference could be calculated and compared with the experimentally observed value.

A different approach was used in the present work. Parametrizations for $F(k^2)$ were tried which satisfy certain criteria. First of all, $F(k^2)$ was chosen so as to make the self-energy integral converge. Secondly, since $2\tau_1 \Delta M$ is negative, one must have $\mathrm{Re}\Sigma(M^2)/\mathrm{Im}\Sigma(M^2)$ positive, and since $\mathrm{Im}\Sigma(M^2)$ is positive, $\mathrm{Re}\Sigma(M^2)$ must also be positive. Inspection of the integral for $\mathrm{Re}\Sigma(M^2)$ shows that the range of integration to the left of $k^2 = \frac{M^2}{4} - m^2$ gives a negative contribution to the integral, while the range to the right of this point gives a positive contribution. Therefore, there is a limit to the rapidity with which $|F(k^2)|$ may be allowed to drop with increasing k^2 and still yield a positive value for $\mathrm{Re}\Sigma(M^2)$. A third condition imposed on the parametrizations of $F(k^2)$ was that they be normalized at the threshold for the $K_1^0 \rightarrow 2\pi$ process; that is, at $k^2 = 0$.

By a procedure to be outlined below, the Omnès equation is inverted to give $\delta(k^2)$ in terms of an integral involving $F(k^2)$. Thus, a knowledge of $F(k^2)$ for all values of k^2 determines $\delta(k^2)$. Such a procedure, whose advantage over the conventional procedure may not be too apparent in

the context of the problem of the $K_1^0 - K_2^0$ mass difference, becomes very effective in the case of the electromagnetic form factor of the pion. With the $e^+ + e^- \rightarrow \pi^+ + \pi^-$ experiments already feasible, one is beginning to know the electromagnetic form factor of the pion. The direct $\pi^+ + \pi^- \rightarrow \pi^+ + \pi^-$ scattering experiments are not likely to take place in the near future. Thus, with the knowledge of the electromagnetic form factor for all momentum transfers, the p-wave pion-pion phase shifts can be determined by the present procedure.

The inversion of the Omnès equation is done by first rewriting it in terms of the variable $s = 4(k^2 + m^2)$ and defining the functions

$$g(s) \equiv \frac{\ln|F(s)|}{(s - 4m^2)}$$

and

$$f(s) \equiv \frac{\delta(s)}{(s - 4m^2)} .$$

Then the integral equation becomes

$$g(s) = \frac{P}{\pi}\int_{4m^2}^{\infty} \frac{f(s')ds'}{s' - s} . \qquad \text{(II.5)}$$

This equation can be converted into the airfoil equation$^{(10)}$, the solution of which is known. The final result in terms

of the variables s and s' is then(11)

$$(s-4m^2)^{\frac{1}{2}}f(s) = \frac{-s}{\pi}\, P\int_{4m^2}^{\infty} \frac{(s'-4m^2)^{\frac{1}{2}}g(s')ds'}{s'(s'-s)} + \frac{2m}{\pi}\int_{4m^2}^{\infty} \frac{f(s')ds'}{s'} \; . \tag{II.6}$$

Because f(s) still appears inside an integral, the equation does not seem to be solved. However, if one calls the first term on the right-hand side h(s) and the second term C, the equation has the special property that

$$\frac{2m}{\pi}\int_{4m^2}^{\infty} \frac{f(s')ds'}{s'} \equiv \frac{2m}{\pi}\int_{4m^2}^{\infty} \frac{h(s') + C}{s'(s'-4m^2)^{\frac{1}{2}}}\, ds' = C \quad ,$$

so that the constant C is arbitrary, and must be determined by some suitable boundary condition. This arbitrariness of C depends only on g(s) being such that the integral involved in h(s) converges. Now,

$$(s-4m^2)^{\frac{1}{2}}f(s) \equiv \frac{\delta(s)}{(s-4m^2)^{\frac{1}{2}}} \quad ,$$

so that if $\delta(s)$ increases more slowly than $s^{\frac{1}{2}}$, the boundary condition is

$$\lim_{s \to \infty}(s-4m^2)^{\frac{1}{2}}f(s) = 0 \quad , \tag{II.7}$$

and hence, the value of C is determined.

Several parametrizations for the K-π-π form fac-

tor were tried, and, after evaluation of the expression for the self-energy, the parameters were adjusted to yield the experimentally observed value for $2\tau_1\Delta M$. The inverted Omnès equation then gave the explicit forms for the I=0, s-wave π-π phase shifts.

The integrals in the calculation of the self-energy can be done analytically (Appendix A), but they are, in general, quite tedious. In addition to this, in the case of the two-parameter parametrizations, the final explicit expressions for the self-energy are extremely complicated, and are, in fact, inseparable functions of the parameters. Therefore, it was decided to calculate the integrals numerically. The singularity was handled using a procedure outlined in Appendix B.

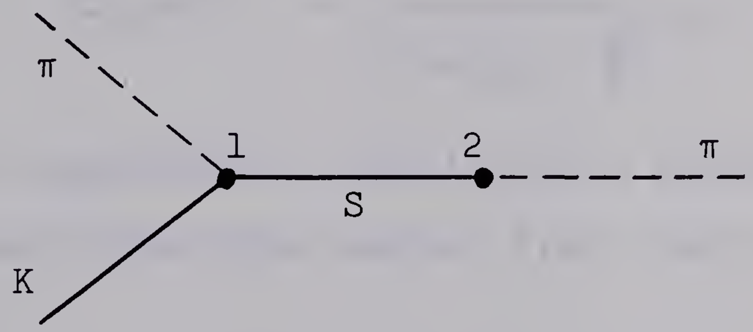
Parametrization I

The first parametrization involves form factors of the kind

$$|F(s)| = \left(\frac{1}{1 + \frac{s-4m^2}{s_0}} \right)^n . \qquad \text{(II.8)}$$

Observe that F(s) has the correct general energy behaviour. It is normalized at threshold ($s=4m^2$), and approaches zero sufficiently quickly in the high energy limit to make the self-energy integral (II.1) converge. Whether it converges too rapidly to yield the correct mass difference can only be determined after the explicit evaluation of the integral (II.1).

The physical justification for this type of parametrization is that for n=1, this form factor represents a particular dynamical model. To see this, first consider the process represented by the Feynman diagram:



where vertex (1) is a strong vertex and vertex (2) is weak, making the whole process first order in the weak interaction. S is a virtual particle of mass, say, M_S, and must be a scalar boson to conserve parity and spin at the strong vertex. In the centre-of-mass of K, each pion has energy $\frac{M}{2}$ and momentum $|\vec{k}|$. If the diagram is thought of as one half of the kaon self-energy diagram, the pions are virtual and therefore the momentum $|\vec{k}|$ is arbitrary. Vertex (2) is a two-particle vertex so that the particle S has the same

[illegible]

[illegible]

four-momentum as the pion. Call this four-momentum p. Then, $p = (\frac{M}{2}, \vec{k})$, which means that the propagator for S is

$$\frac{1}{p^2 - M_s^2} = \frac{-1}{k^2 + M_s^2 - \frac{M^2}{4}} .$$

Now, the propagated particle S is responsible for the structure (i.e. energy dependence) of the K-π-π vertex. A form factor, by definition, is just a function which contains the energy dependence of some single-vertex process. Therefore, the "form factor" for the diagram shown is

$$|F'(k^2)| = \frac{\alpha}{k^2 + M_s^2 - \frac{M^2}{4}} ,$$

where α is the appropriate normalization constant. Notice that the parametrized form factor, F(s), can be written in the form

$$|F(k^2)| = \frac{s_0/4}{k^2 + \frac{s_0}{4}}$$

when n=1. Obviously, $F(k^2)$ corresponds to $F'(k^2)$ with $\frac{s_0}{4} = M_s^2 - \frac{M^2}{4}$, or if $M_s^2 >> \frac{M^2}{4}$, with $\frac{s_0}{4} \simeq M_s^2$. This shows that this parametrization corresponds to a physical model in which a very massive scalar boson ($M_s = \sqrt{s_0/4} >> \frac{M}{2}$) is produced in an intermediate state between the kaon and one of the pions.

In order to observe the difference in behaviour of the phase shifts for form factors with different rates of convergence in the high energy limit, different values of n were tried. The parameter s_0 was adjusted to give the correct value for $2\tau_1\Delta M$ for the four cases $n = \frac{1}{2}$, 1, $\frac{3}{2}$, and 2. The results are given in the following table.

n	$s_0 \begin{pmatrix}\text{in units}\\ \text{of } m^2\end{pmatrix}$
0.5	101
1	288
1.5	480
2	672

Table 1. Values of n and s_0 in Parametrization I which yield $2\tau_1\Delta M = -1.0$

These results were obtained numerically; however, as a matter of interest, an analytic method for the evaluation of $\mathrm{Re}\Sigma(M^2)$ is given in Appendix A.1 .

The solution of the Omnès equation for this parametrization is (Appendix C.1)

$$\delta(s) \quad = \quad 2n \arctan\left(\frac{s-4m^2}{s_0}\right)^{\frac{1}{2}}. \qquad \text{(II.9)}$$

[illegible]

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
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$$[illegible]$$

One sees that the above phase shift exhibits the correct threshold behaviour; that is, it approaches zero as $\sqrt{s-4m^2}$. From the low-energy scattering length formula,

$$\tan \delta_0 \simeq \delta_0 \simeq ka_0 = \frac{a_0}{2}\sqrt{s-4m^2} , \qquad (II.10)$$

where the subscripts refer to the $\ell=0$ state, the I=0, s-wave scattering lengths can easily be determined for each of the pairs of n and s_0. Table 2 gives the scattering lengths for the four values of n.

n	$a_0 \begin{pmatrix}\text{in units}\\ \text{of } 1/m\end{pmatrix}$
0.5	0.20
1	0.24
1.5	0.27
2	0.31

Table 2. s-wave, I=0 π-π scattering lengths for different values of n in Parametrization I.

Figure 1 shows the energy dependence of the phase shifts for the four cases. The plots are phase shift in degrees versus centre-of-mass energy ($\sqrt{s}$) in Mev.

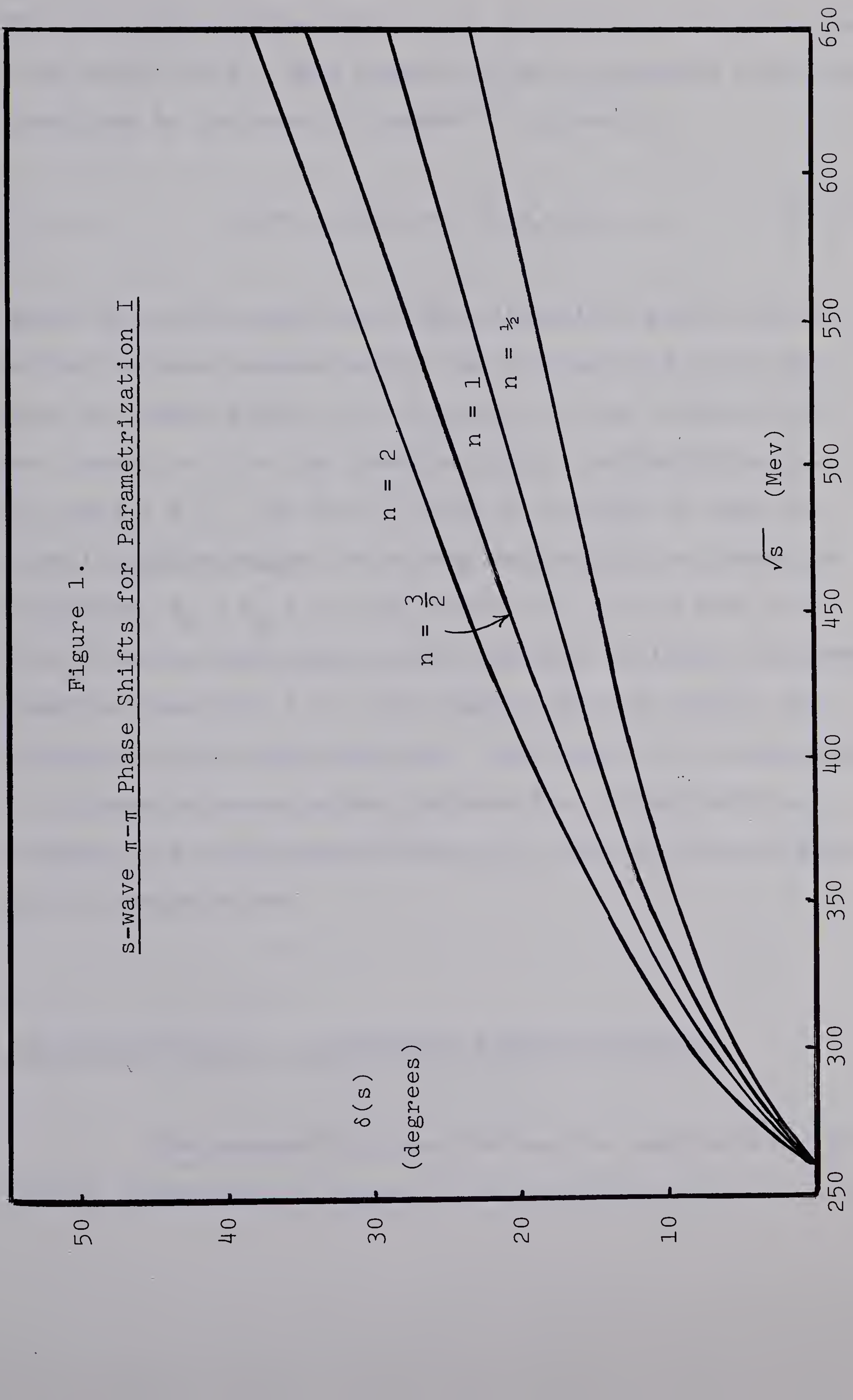


Figure 1.

s-wave π-π Phase Shifts for Parametrization I

The four phase shifts can be seen to tend to $n\pi$ for the various values of n. This behaviour can be compared with that predicted by Levinson's theorem$^{(12)}$, given by

$$\delta(\infty) - \delta(0) = \pi(N_b - N_a) , \qquad (II.11)$$

where N_b is the number of stable elementary particles involved in some process before the interactions which produce the phase shifts (in this case, strong interactions) are turned on. In the dynamical model corresponding to the case $n = 1$, the K-S-π vertex is strong, so that the kaon is stable before the strong interaction is turned on. Therefore, $N_b - N_a = 1$, and, since $\delta(0)$ can be seen to be zero from the scattering length formula, Levinson's theorem predicts that $\delta(\infty) = \pi$, which agrees with the result obtained from the Omnès equation. The case $n = 2$ corresponds to a dynamical model which includes two strong vertices; however, the half-integer values of n lead to no such physical interpretation.

Parametrization II (Resonance Parametrization)

For resonant π-π scattering, the appropriate form factor, given here in terms of the variable s, is

$$F(s) = \frac{s_r - 4m^2}{(s_r - s) - 2i\gamma\sqrt{s-4m^2}} . \qquad \text{(II.12)}$$

s_r is the square of the mass, M_r, of the resonance and γ is related to Γ, the full width at half maximum of $|F(s)|^2$ by an expression which is quite complicated in general, but which in the application to the problem of the $K_1^0 - K_2^0$ mass difference, is quite adequately approximated by (Appendix D)

$$\Gamma \simeq 2\gamma\left(\frac{s_r - 4m^2}{s_r}\right)^{\frac{1}{2}}\left(1 - \frac{\gamma}{2\sqrt{s_r}}\right) , \qquad \text{(II.13)}$$

where only terms up to first order in $\gamma/\sqrt{s_r}$ have been retained.

From the form of $\mathrm{Re}\Sigma(M^2)$, it is evident that the range of integration $0 < k^2 < \frac{M^2}{4} - m^2$ gives a negative contribution to the integral. This corresponds to $s < M^2$ (i.e. pion-pion centre-of-mass energies less than the kaon mass). The range of integration for energies above the kaon mass gives a positive contribution to the integral. Therefore, resonances below the kaon mass will tend to make $\mathrm{Re}\Sigma(M^2)$ negative and since

$$2\tau_1\Delta M = \frac{-\mathrm{Re}\Sigma(M^2)}{\mathrm{Im}\Sigma(M^2)} ,$$

this will lead to positive values of ΔM. ΔM is known to be

negative, and therefore one expects that only resonances above the kaon mass can lead to the correct value for ΔM. (It is not quite obvious that a broad resonance peak below the kaon mass will not yield a negative mass difference; however, calculations were done using resonances below the kaon mass and it was found impossible to produce the correct mass difference for these cases.)

Pairs of γ and s_r were found which yield $2\tau_1 \Delta M = -1.0$. The corresponding values of the full width, Γ, and the resonance mass, $M_r = \sqrt{s_r}$, are listed in Table 3 together with the scattering lengths derived from the low-energy limits of the phase shifts, $\delta(s)$, which are the solutions of the Omnès equation.

M_r(Mev)	Γ (Mev)	a_0(1/m)
510	24	0.04
550	97	0.13
600	184	0.19
700	357	0.25

Table 3. Pairs of M_r and Γ which yield $2\tau_1 \Delta M = -1.0$, with corresponding π-π scattering lengths.

The resonance is seen to be quite narrow for values of M_r near the kaon mass, but as M_r increases above M, the full width Γ increases rapidly from a value of 24 Mev at M_r = 510 Mev to a value of 357 Mev at M_r = 700 Mev.

The phase shift obtained from the solution of the Omnès equation for the resonance parametrization is given by (Appendix C.2)

$$\delta(s) = \arctan\left(\frac{\gamma k}{k_r^2 - k^2} \right) = \arctan\left(\frac{2\gamma\sqrt{s-4m^2}}{s_r - s} \right) . \qquad \text{(II.14)}$$

Figure 2 displays the phase shift as a function of energy ($\sqrt{s}$) for the four pairs of Γ and M_r. All of the phase shifts are seen to approach 180° at high energies, in agreement with Levinson's theorem (The resonance is considered to be a stable "elementary" particle in the absence of strong interactions.). An interesting property of these phase shifts is that their values at the kaon mass (≈500 Mev) all lie within a range of about 10° (between 33° and 44°), so that if an experimental determination of $\delta(M^2)$ fixes its value outside this limited range, it will be difficult to describe the π-π scattering process by means of a resonance parametrization. For example, to obtain a value for $\delta(M^2)$ above 45°, an extremely narrow resonance near the kaon mass will be required, and to obtain a value for $\delta(M^2)$ below 30°, a very broad resonance somewhere above

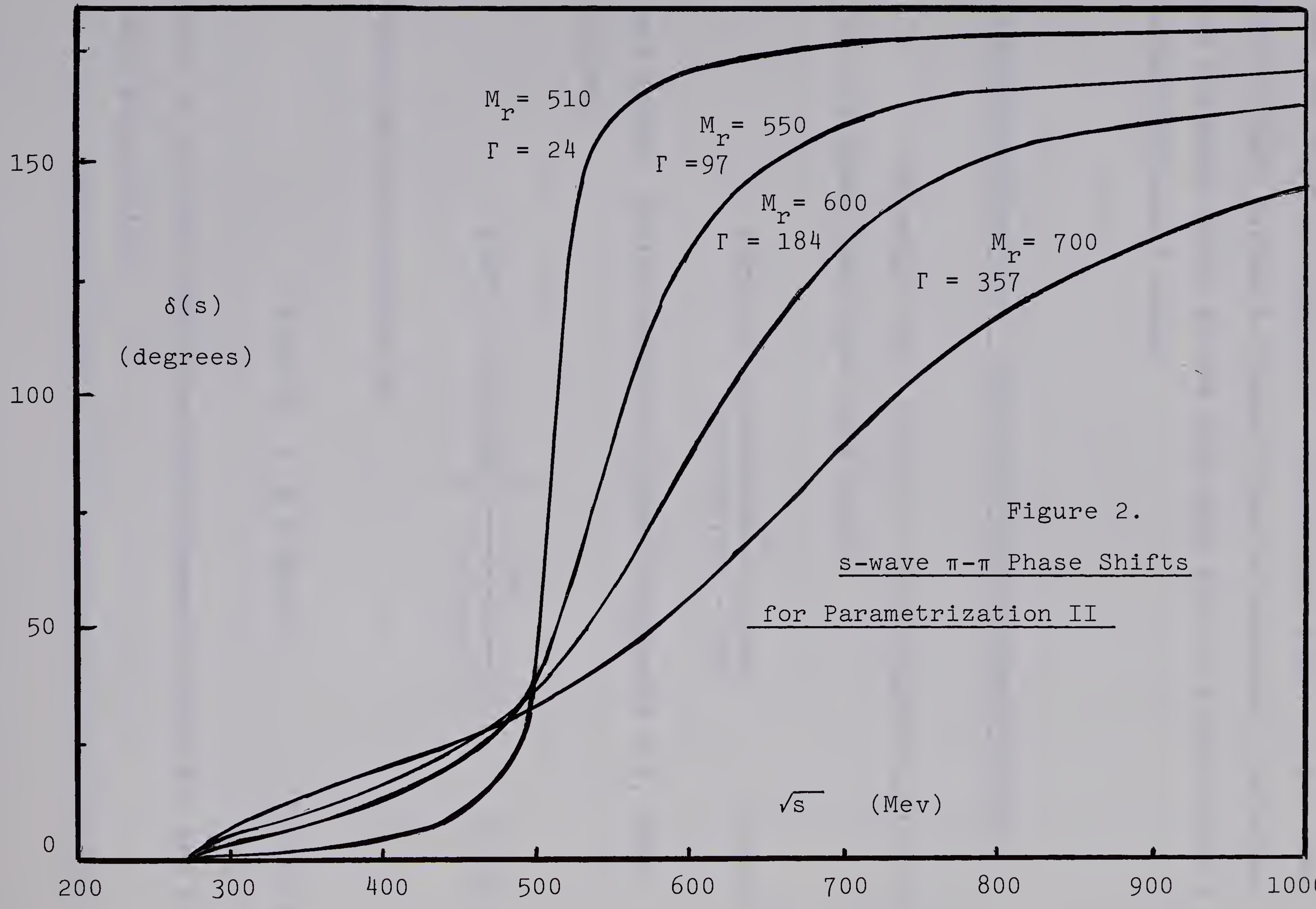
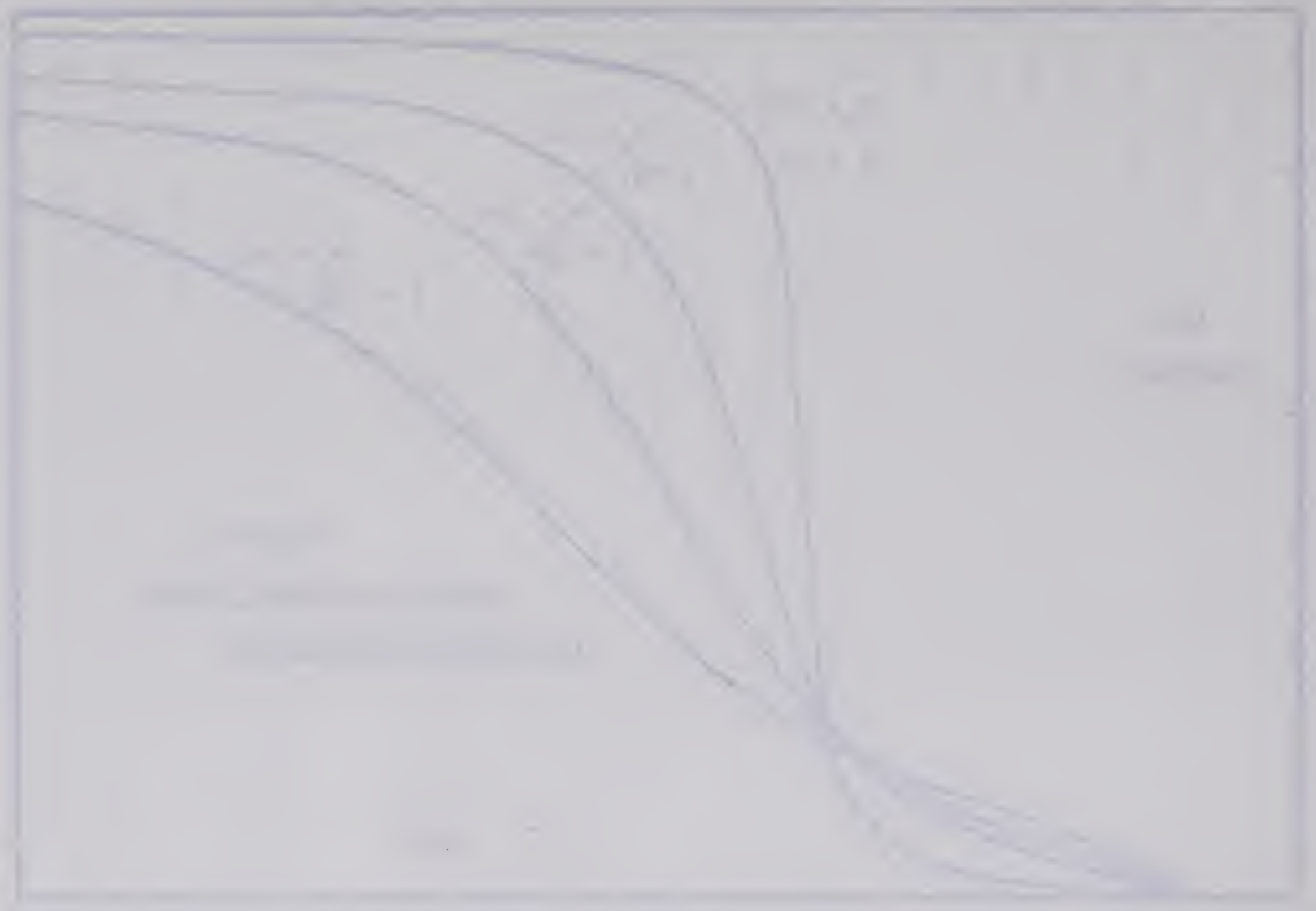


Figure 2.

<u>s-wave π-π Phase Shifts</u>

<u>for Parametrization II</u>



800 Mev will be required. On the other hand, if the two-pion s-wave system is indeed found to resonate, the value of the phase shift at the kaon mass will be fixed within a relatively small range.

Parametrization III (Effective Range Parametrization)

To obtain the effective range parametrization for the π-π phase shift, the form factor

$$F(k^2) = \frac{1}{(1 + \tfrac{1}{2}ar_0k^2) - iak} \qquad \text{(II.15)}$$

was tried. By the method outlined in Appendix C.2, this yields

$$\delta(k^2) = \arctan\left(\frac{ka}{1 + \tfrac{1}{2}r_0ak^2}\right), \qquad \text{(II.16)}$$

which can be rewritten as

$$k \cot \delta = \frac{1}{a} + \tfrac{1}{2}r_0k^2. \qquad \text{(II.17)}$$

This is just the effective range parametrization for the s-wave phase shift.

Pairs of a and r_0 were found which gave the cor-

rect value for $2\tau_1\Delta M$. These pairs included positive and negative values of both a and r_0. The solutions were found to occur in four separate branches, two of which are the reflections of the other two through the origin. This symmetry is expected, since the quantity $|F(k^2)|^2$ is used in the calculation of $2\tau_1\Delta M$ and, from the form of Equation (II.15), it is seen to be left unchanged when both a and r_0 change sign. The data for two of the characteristic branches is tabulated in Table 4 and the corresponding graph of r_0 versus a is displayed in Figure 3. Notice that only the range $-0.275 < a < 0.275$ will yield the correct $2\tau_1\Delta M$.

It can be shown (Appendix E) that for $a < 0$ and $r_0 > 0$, the form factor as given describes a system which has a bound state. Now, no two-pion bound state is known to exist, and therefore these values of a and r_0 are unrealistic. Furthermore, if a bound state does exist, the relation (II.4) between the form factor and the phase shift no longer holds$^{(13)}$. Therefore, the values of a and r_0 corresponding to the left branch of Figure 3 will not be considered further.

If $a > 0$ and $r_0 < 0$, the phase shift (II.16) becomes the resonance phase shift which has been analyzed in part II of this chapter. This leaves only the solutions with the same sign for a and r_0 to be considered.

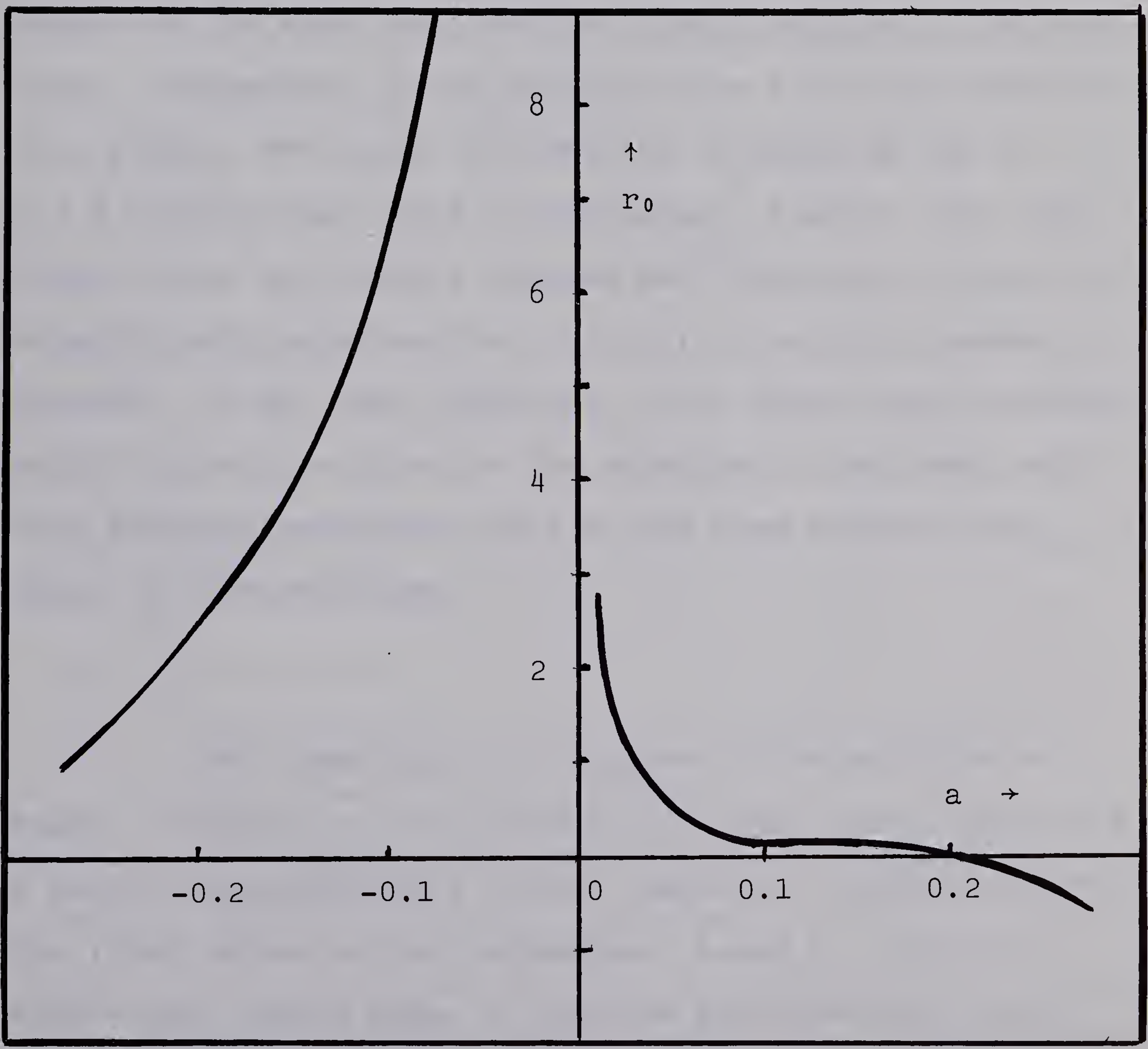
From potential scattering theory, the effective

a	-.27	-.20	-.10	.01	.05	.10	.20	.275
r_0	.95	2.48	6.90	2.77	.52	.21	0	-.51

Table 4.

<u>Pairs of a and r_0 Leading to $2\tau_1\Delta M = -1.0$</u>

Figure 3.



range r_0 is defined by$^{(14)}$

$$\tfrac{1}{2}r_0 = \int_0^\infty (u_0^2 - u^2)\,dr \quad , \qquad \text{(II.18)}$$

where u is the exact solution of the Schrödinger equation and u_0 is the free-particle solution, which is equal to u outside the range of the forces. u is zero at r=0 but u_0 is not, so that in simple cases where the forces are short-range and the wave functions are slowly varying, r_0 is positive. Therefore, if one knew that the π-π forces were in fact simple, one would be justified in ignoring the $a < 0$, $r_0 < 0$ solution for the π-π scattering. However, for long-range forces and quickly varying wave functions, r_0 can be negative and therefore the $a < 0$, $r_0 < 0$ solution cannot be dropped. In any case, Equation (II.6) shows that the phase shift for this solution is the negative of the phase shift with positive parameters, and so the same analysis will apply to both solutions.

The form factor is expected to be modified at higher energies by the inclusion of higher order terms in k. A faster-converging form factor lowers the value of $\mathrm{Re}\Sigma(M^2)$ for fixed values of the parameters, a and r_0, since the high-energy region makes a positive contribution to the

integral in $\mathrm{Re}\Sigma(M^2)$. In the process of fitting a and r_0 to obtain the correct mass difference, it was found that, for a fixed value of a, $\mathrm{Re}\Sigma(M^2)$ was increased when r_0 was decreased. Thus, the effect of the higher order terms would be to decrease the effective range in relation to the scattering length. Of course, if the higher order terms cause the form factor to converge more slowly in the high energy region, the opposite effect would be observed.

Notice that for $r_0 = 0$, the effective range form factor reduces to the simple form

$$F(k^2) = \frac{1}{1 + iak} .$$

This corresponds to the first parametrization with $n = \frac{1}{2}$ and $a^2 = 4/s_0$. The value of $s_0 \simeq 100\ m^2$ for this case agrees with the value of $a = 0.2$ found for the effective range parametrization, and the phase shifts in both cases tend to 90° in the high energy limit. In the effective range parametrization, this is the only value of r_0 which leads to this asymptotic behaviour; the phase shifts for $r_0 > 0$ tend to zero at high energies.

The energy dependence of the phase shifts for several pairs of a and r_0 is displayed in Figure 4. The phase shifts are seen to be very slowly varying functions of the pion-pion centre-of-mass energy. For example, the

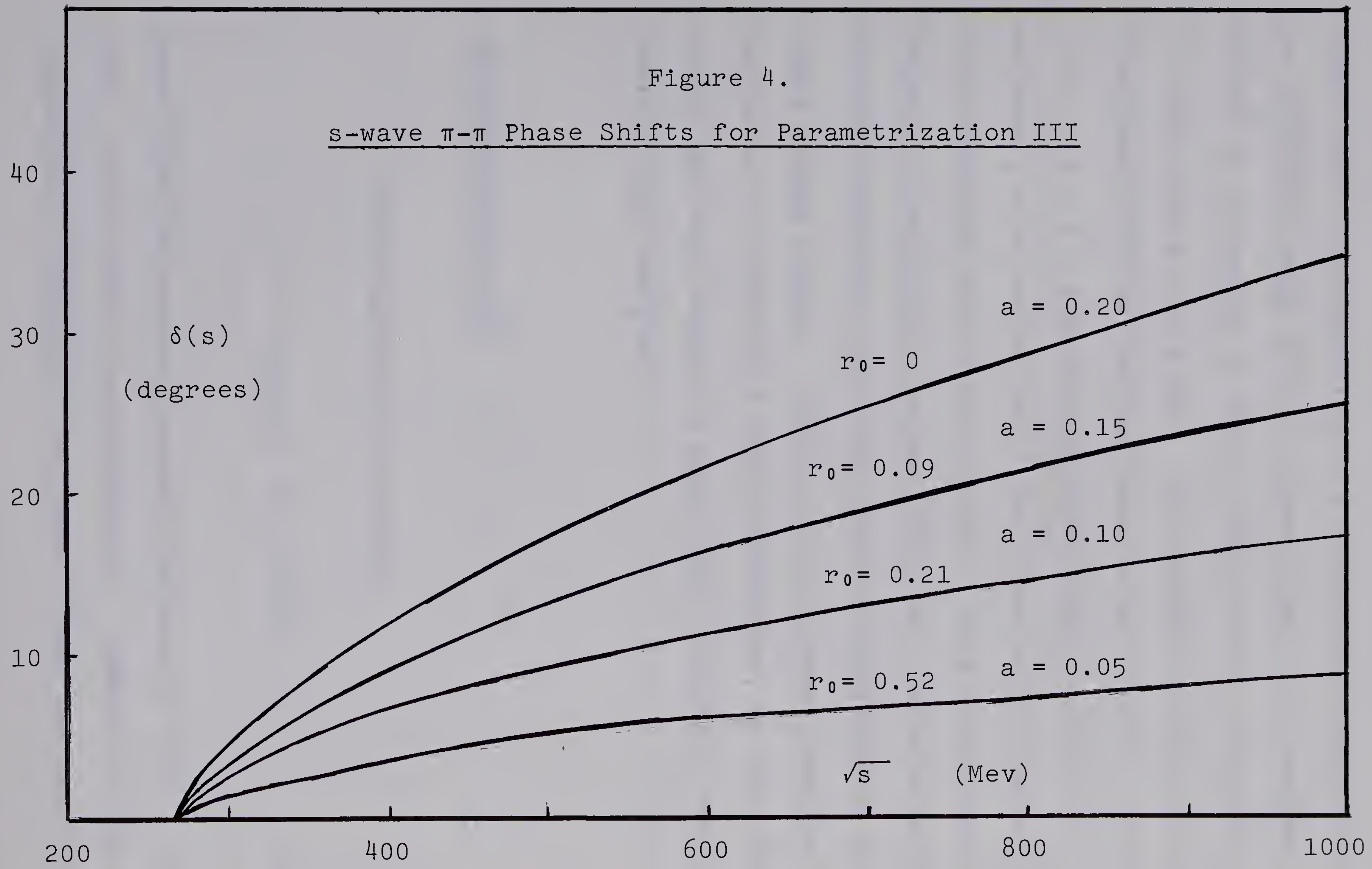
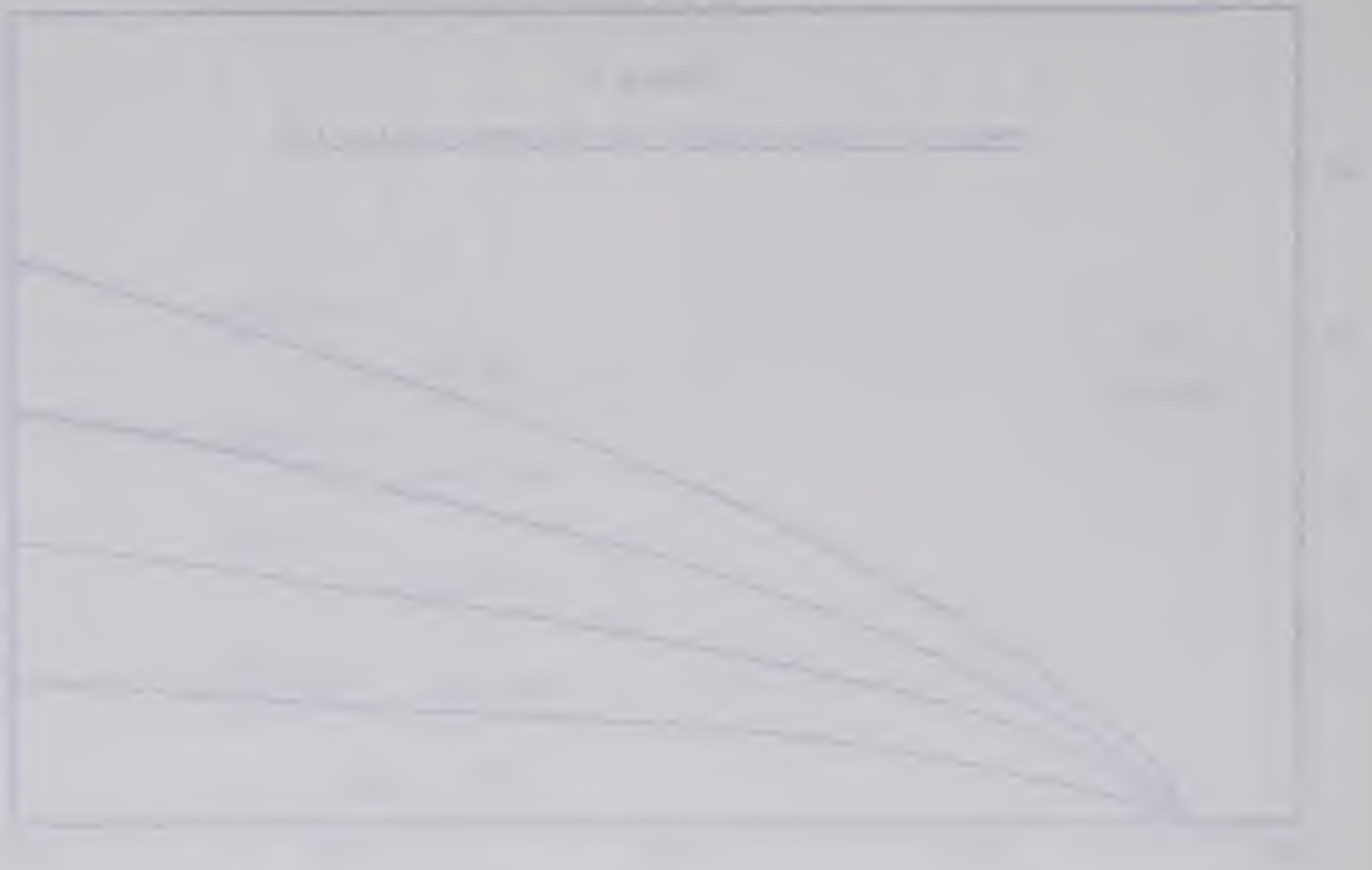


Figure 4.

s-wave π-π Phase Shifts for Parametrization III



phase shift corresponding to a scattering length of 0.1 and an effective range of 0.21 rises slowly to a maximum value of 26° at an energy of about 2600 Mev and decreases slowly to zero beyond this point. The maximum possible value of the phase shift at the kaon mass occurs for the case $a = 0.2$, $r_0 = 0$, and this maximum value is only 17°. This extremely slow variation of the phase shifts reflects the slow variation of the form factor. This form factor represents a scattering process with very little structure, and the scattering amplitude varies only slightly over large energy ranges. However, this parametrization is expected to be valid only for low energies, so that the phase shifts may well exhibit a more rapid variation even at moderately high energies.

Parametrization IV

Form factors of the form

$$|F(s)| = e^{-sG(s)} \tag{II.19}$$

were tried for several parametrizations of G(s). These were, on the whole, unsatisfactory. However, it is still worthwhile to calculate the phase shifts for the different

forms of G(s) in order to obtain a more general knowledge of the relation between the form factor and the phase shifts.

(a) The parametrization

$$G(s) = \alpha \quad ,$$

where α is a constant, leads to a phase shift of the form (Appendix C.4.a)

$$\delta(s) = \beta\ (\ s - 4m^2\ )^{\frac{1}{2}} \quad . \qquad \text{(II.20)}$$

Because δ(s) is only bounded in this way, the boundary condition

$$\lim_{s\to\infty} \frac{\delta(s)}{(s - 4m^2)^{\frac{1}{2}}} = 0$$

cannot be imposed and hence β cannot be determined in terms of α. In addition to this, δ(s) becomes infinite in the high energy limit, which is totally unrealistic. Thus, this parametrization is not useful in the application to π-π scattering.

(b) The parametrization

$$G(s) = \frac{\alpha}{(s - 4m^2)^{\frac{1}{2}}}$$

leads to a phase shift of the form (Appendix C.4.b)

$$\delta(s) = \frac{\alpha}{\pi} (s - 4m^2)^{\frac{1}{2}} \ln\left(\frac{s - 4m^2}{4m^2}\right) . \qquad \text{(II.21)}$$

This phase shift behaves even more unrealistically than the previous one. It diverges as $s^{\frac{1}{2}}\ln s$ at high energies, and its behaviour at threshold is like $k \ln k$. While one may be prepared to accept phase shifts with unrealistic high energy behaviour which appear reasonable at low energies, one certainly cannot accept phase shifts which exhibit unrealistic behaviour at both the high and low energy limits.

(c) The parametrization

$$G(s) = \frac{\alpha}{\sqrt{s}}$$

leads to a phase shift of the form

$$\delta(s) = \frac{\alpha}{\pi} \frac{(s - 4m^2)}{s^{\frac{1}{2}}} \ln\left(\frac{s^{\frac{1}{2}} + (s - 4m^2)^{\frac{1}{2}}}{s^{\frac{1}{2}} - (s - 4m^2)^{\frac{1}{2}}}\right) .$$

This phase shift diverges at s→∞ like $s^{\frac{1}{2}}/\ln s^{\frac{1}{2}}$. The

threshold behaviour is found to be like k^3, which again is unrealistic for the s-wave phase shift. Notice that although G(s) has the same asymptotic behaviour for both parametrizations (b) and (c), the asymptotic behaviour of the phase shifts is quite different. The reason for this is that the solution for $\delta(s)$ involves an integration of G(s) over all energies, and the low-energy region can produce effects at high energies. Thus, it is not possible to guess the form of the phase shift simply from a study of the asymptotic behaviour of the form factor; one must know the form factor at all energies to determine the phase shift.

This treatment of pion-pion scattering leads to several interesting observations. It can be seen that form factors which are slowly varying functions of energy also produce phase shifts which are slowly varying. For example, for the first parametrization values of $n = \frac{1}{2}$ and $s_0 = 101$ m^2 produce a scattering length of 0.20 and a phase shift which rises from 0° at threshold to only 17° at the kaon mass. Larger values of n with the corresponding values of s_0 consistent with the observed $K_1^0 - K_2^0$ mass difference yield larger scattering lengths and higher phase shifts at the kaon mass. The values $n = 2$ and $s_0 = 672$ m^2 lead to a

scattering length equal to 0.31 and a phase shift of 27° at the kaon mass. The high energy behaviour of the phase shifts for the first parametrization is somewhat unrealistic - the phase shift for a given n tends to $n\pi$. However, they approach these limits sufficiently slowly so that in the low energy region the phase shifts are not too sensitive to changes in the parameters, and the behaviour near threshold is nearly the same for all values of n.

The effective range parametrization is a parametrization which produces even more slowly varying phase shifts. The positive scattering lengths in this parametrization lie between 0 and 0.2, and lead to phase shifts which rise to a maximum of only 17° at the kaon mass. This maximum phase shift corresponds to a scattering length of 0.2 and an effective range of 0. For these values of the parameters, the effective range parametrization reduces to the first parametrization, and of course produces the same phase shift and scattering length. Smaller values of a lead to more slowly varying phase shifts. For example, a scattering length of 0.1 with the corresponding effective range of 0.21 leads to a phase shift of about 9° at the kaon mass. Therefore, if experiments show that the s-wave pion-pion phase shift is above 17° , the effective range parametrization as given , even though it leads to plausible threshold and high-energy behaviour, will be unable

to satisfactorily describe pion-pion scattering.

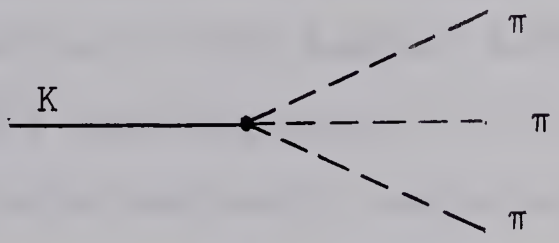
The resonance form factor is one which is more quickly varying than the form factors discussed above. Hence, this parametrization is capable of producing larger phase shifts. As mentioned before, this parametrization requires that any di-pion s-wave resonance lies above the kaon mass in order to account for the observed $K_1^0 - K_2^0$ mass difference. As the position of the resonance rises above the kaon mass, the corresponding width increases rapidly. An interesting feature of this parametrization is that, even though the different phase shifts arising from the different pairs of resonance mass and width are rather widely separated at most energies, the phase shifts at the kaon mass lie within a relatively small range, about 10° for resonances between 510 Mev and 700 Mev, and it requires extreme values of the parameters to bring the kaon mass phase shift out of this range. For narrow resonances, this parametrization leads to small scattering lengths ($a = 0.04$ for $\Gamma = 24$ Mev and $M_r = 510$ Mev), more rapidly varying phase shifts, and a higher value for δ at the kaon mass ($\delta(M^2) = 44^{\circ}$ for this case). Wide resonances, on the other hand, lead to larger scattering lengths ($a = 0.25$ for $\Gamma =$ 356 Mev and $M_r = 700$ Mev), more uniform variation of the phase shifts with energy, and lower values of the phase shift at the kaon mass ($\delta(M^2) = 34^{\circ}$ for $\Gamma = 356$ Mev, $M_r =$

700 Mev).

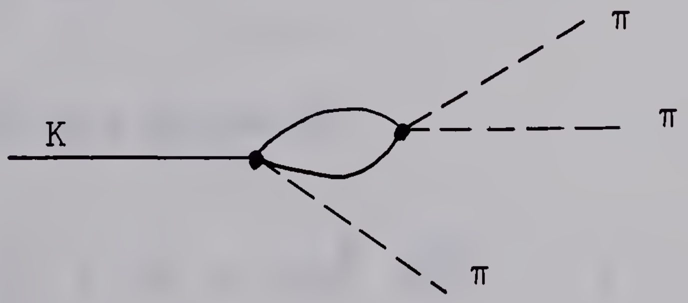
It is evident that, for these three parametrizations, it is difficult to obtain large scattering lengths without assigning extreme values to the parameters and, in fact, the effective range parametrization can never lead to a scattering length above 0.275. Similarly, one cannot obtain values for the phase shifts at the kaon mass higher than about 45° without, for instance, having n considerably greater than 2 in the first parametrization, or having Γ smaller than about 20 Mev in the resonance parametrization. As the parameters approach these extreme values, it becomes increasingly difficult to give a physical interpretation to the process leading to the form factors, so that if these parametrizations do require extreme values of the parameters to fit experimental data, it may be advisable to try to find other form factors which have a simpler physical interpretation.

CHAPTER III K → 3π DECAY

Another process from which one can obtain information on the pion-pion interaction is the weak decay of the kaon into three pions. There are two points of view possible in approaching this problem, as summarized by Khuri and Treiman(15). One is the idea that the structure of the K → 3π decay is contained in the basic vertex, corresponding to a Feynman diagram like



This approach is taken by, for example, H. Abarbanel(16). The other point of view is that final-state interactions between the pions lead to the energy-dependent structure of the matrix element. The corresponding diagram is one like



Here, the final-state interactions are assumed to occur only between pairs of pions, and any interaction involving all three pions at once is neglected. This is the approach which is adopted in the present work.

The basic interaction involves the coupling of the K-meson field to the three pion fields. Since the kaon firld transforms like an isodoublet and the pion field like an isotriplet, isospin is not conserved in this weak decay. If, however, the $|\Delta I| = \frac{1}{2}$ rule is assumed to be valid, there is a procedure by which the weak interaction may be written in an isoscalar form(17). One introduces a fictitious particle called the "spurion", represented by a field S, which has the property of carrying one half-unit of isospin but has no other physical properties. An isovector field is constructed out of the spurion and kaon fields by means of the isotopic spin matrices, τ_1, τ_2, and τ_3(identical with the Pauli spin matrices), and this isovector is coupled to the isovector pion fields to form an isoscalar. The isoscalar interaction Hamiltonian formed in this way is

$$H_W = G\,(\bar{S}\,\vec{\tau}\cdot\vec{\phi}\,K)(\vec{\phi}\cdot\vec{\phi}) \quad , \qquad \text{(III.1)}$$

where the fields are given by

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} \; ; \; \bar{S} = (S_1^* \;\; S_2^*) \; ; \; K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} .$$

$\vec{\phi}$ is the isovector pion field. If one defines the quantities

$$\tau_+ = \frac{\tau_1 + i\tau_2}{\sqrt{2}} \quad ; \quad \tau_- = \frac{\tau_1 - i\tau_2}{\sqrt{2}} \quad ; \quad \tau_0 = \tau_3$$

and

$$\phi_+ = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad ; \quad \phi_- = \frac{\phi_1 - i\phi_2}{\sqrt{2}} \quad ; \quad \phi_0 = \phi_3 \qquad ,$$

it is seen that

$$\vec{\tau}\cdot\vec{\phi} = \tau_+\phi_- + \tau_-\phi_+ + \tau_0\phi_0$$

and

$$\vec{\phi}\cdot\vec{\phi} = \phi_+\phi_- + \phi_-\phi_+ + \phi_0\phi_0 \quad .$$

The subscripts on the pion fields are now identified with the charge of the emitted pions. One also defines the fictitious isovector field Φ by

$$\vec{\Phi} = \bar{S}\,\vec{\tau}\,K \quad ,$$

with

$$\Phi_+ = \bar{S}\tau_+K = \sqrt{2}\,S_1^*K^0$$

$$\Phi_- = \bar{S}\tau_- K = \sqrt{2}\, S_2^* K^+$$

and $$\Phi_0 = \bar{S}\tau_0 K = S_1^* K^+ - S_2^* K^0 \quad .$$

Then the Hamiltonian can be written

$$H_W = G\, (\Phi_+\phi_- + \Phi_-\phi_+ + \Phi_0\phi_0)(\vec{\phi}\cdot\vec{\phi}) \quad .$$

In this work, only the decay of the positively charged kaon will be considered. Then only Φ_- and Φ_0 can contribute to this process. However, notice that since the term $\Phi_0\phi_0$ couples to the neutral combination $\vec{\phi}\cdot\vec{\phi}$, the term containing the charged kaon field does not conserve charge. This arises because the formalism used here conserves isospin without considering charge. All members of an isomultiplet have the same strangeness and baryon number, and since the two members of the spurion isodoublet differ in the third component of isospin by one unit, the Gell-Mann - Nishijima relation

$$Q = I_3 + \frac{B + S}{2} \quad ,$$

where Q is the charge, B is the baryon number, and S is the strangeness of a hadron, shows that the two spurions, S_1 and S_2 , should differ in charge by one unit. The spurion

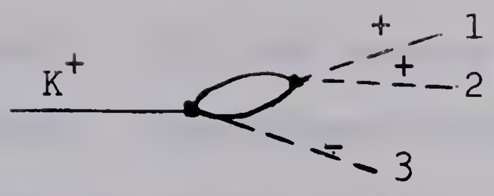
is a fictitious particle and therefore it cannot be assigned any physical property such as charge, and so conservation of isospin will lead to non-conservation of charge and vice-versa. Therefore, disregarding the charge non-conserving term, the only terms contributing to the decay of the charged kaon are

$$H_W^+ = \sqrt{2}\, G\, S_2^* K^+ (\phi_+\phi_- + \phi_-\phi_+ + \phi_0\phi_0) \ . \qquad \text{(III.2)}$$

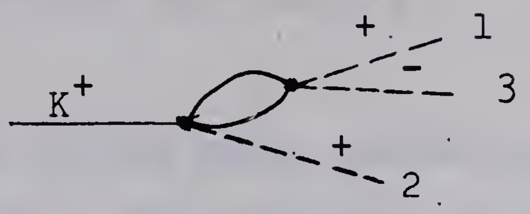
The first two terms correspond to the decay $K^+ \rightarrow \pi^+ + \pi^+ + \pi^-$, known as the " τ " decay mode, and the last term corresponds to the decay $K^+ \rightarrow \pi^+ + \pi^0 + \pi^0$, known as the " τ' " decay mode. Since all three pion-field terms are weighted equally, one immediately observes that if the τ' basic vertex is characterized by a strength, say, g, then the τ vertex is characterized by the strength 2g. The two decay modes will be considered separately.

III.1 τ Decay of the K^+ Meson ($K^+ \rightarrow \pi^+\pi^+\pi^-$)

Assuming that final-state interactions occur between pairs of pions , the diagrams contributing to the second order matrix element are

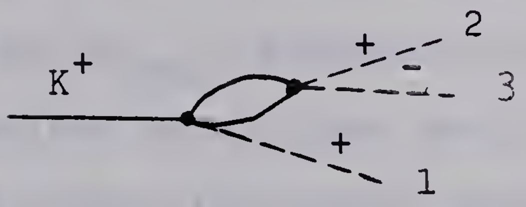


1(a)



1(b)

and



1(c)

where the pions are labelled 1, 2, and 3 as shown, with 3 designating the "odd" pion (in this case, the negative pion). Each of these diagrams must be summed over all possible intermediate states. For the first diagram, the only possible intermediate state is the one with two positively charged pions (by conservation of charge). In the second and third diagrams, however, the intermediate state is a neutral two-pion state and therefore can be either $\pi^+\pi^-$ or $\pi^0\pi^0$. Notice that if the intermediate state contains two charged pions, the basic weak vertex is the τ vertex, whereas if the intermediate state contains two uncharged pions, the basic vertex is the τ' vertex. The K-meson is a spinless particle so that the two pions may interact in the final state in either an s-state or a p-state, with the third pion in an angular momentum state which makes the total angular momentum zero (d-waves are suppressed by

a centrifugal barrier). Only the s-wave pions will be considered in this section. The p-wave analysis for both the τ and τ' modes is done in Section III.3 .

The effect of the basic vertex is given by g or 2g as shown previously. The effect of the pion-pion vertex will be denoted by $iT_{\alpha\beta\gamma\delta}$, where $\alpha\beta$ are the charge indices of the intermediate state pions and $\gamma\delta$ are the charge indices of the final-state pions which have taken part in the final-state interaction. For example, $T_{++,++}$ represents the scattering of two positively charged pions. The effect of propagating the intermediate state pions between the vertices will be denoted by I(i), where i is the number of the pion not taking part in the final-state interaction. Then, after summing over intermediate states in diagram 1(a) (only one intermediate state in this case), the contribution from this diagram to the τ matrix element is given by (redefining $I \rightarrow iI$, for reasons given later)

$$2g\ I(3)\ T_{++,++} \quad .$$

In diagram 1(b), the possible intermediate states are (+-), (-+), or (00). The contribution from this diagram is then

$$2\times 2g\ I(2)\ T_{+-,+-} \ + \ g\ I(2)\ T_{00,+-}$$

where the extra factor of 2 comes from the two possible charged pion states, (+-) or (-+). Similarly, the contribution of diagram 1(c) is

$$2\times 2g\ I(1)\ T_{+-,+-} + g\ I(1) T_{00,+-}$$

By extended Bose statistics, the two interacting pions must be either in an I = 0 or I = 2 isospin state, since their relative angular momentum is 0. A Clebsch-Gordan analysis shows that(18)

$$T_{++,++} = A_2$$

$$T_{+-,+-} = \frac{1}{3}A_0 + \frac{1}{6}A_2$$

and

$$T_{00,+-} = \frac{1}{3}A_0 - \frac{1}{3}A_2 \quad ,$$

where A_0 and A_2 are the scattering amplitudes in the I=0 and I=2 isospin states, respectively. In the zero range approximation

$$A_{0,2} \simeq 32\pi\ a_{0,2} \quad ,$$

where $a_{0,2}$ are the I=0 and I=2 scattering lengths. Substituting for the T's and collecting terms, one finds the final-state interaction correction term to the τ matrix

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

element to be, apart from a normalization constant which is the same for the first-order basic vertex,

$$M'_{\tau,s} = 32\pi\, g\, a_0\left(\frac{5}{3}I(1) + \frac{5}{3}I(2)\right)$$
$$+\ 32\pi\, g\, a_2\left(\frac{1}{3}I(1) + \frac{1}{3}I(2) + 2I(3)\right) \quad , \tag{III.3}$$

where the prime indicates the correction term and s shows that this is the correction term when the pions interact in the s-state. The quantity I(i) is an integral involving two pion propagators and two four-momentum-conserving δ-functions. Now, each δ-function brings with it a factor $(2\pi)^4$ and each propagator contributes a factor $i/(2\pi)^4$. However, it has already been assumed that the basic vertex is represented by the quantity g , so that g already contains one of the factors $(2\pi)^4$ arising from the δ-function at the basic vertex. Then, letting i, j, and k represent the number labels on the three real pions, one has

$$I(i) = \frac{-i}{2(2\pi)^4}\iint d^4k_1 d^4k_2 \frac{\delta^4(K-p_i-k_1-k_2)\delta^4(k_1+k_2-p_j-p_k)}{(k_1^2 - m^2)(k_2^2 - m^2)} \quad ,$$

where K, k_1, k_2, p_1, p_2 are the four-momenta of the kaon and the intermediate and final-state pions. The factor ½ arises because of the multiplicity of the diagrams. The first integration removes one of the δ-functions, leaving

an over-all energy-momentum conserving δ-function which will be considered part of the normalization factor which has been omitted. Then, dropping the subscripts on the k's and defining $P_i \equiv p_j + p_k$, one finds that

$$I(i) = \frac{-i}{2(2\pi)^4} \int \frac{d^4k}{(k^2 - m^2)\left((k - P_i)^2 - m^2\right)} \,. \qquad (III.4)$$

This expression is Lorentz invariant and can therefore be evaluated in any convenient reference frame. In this case, the frame corresponding to the centre of mass of the two interacting pions was found to be most convenient. In this frame, $\vec{P}_i = \vec{p}_j + \vec{p}_k = 0$, and $p_j^0 = p_k^0 = \frac{1}{2}P_i^0$. For convenience, one writes these quantities in the two-pion centre of mass as $|\vec{p}_j| = |\vec{p}_k| \equiv k_i$, and $p_j^0 = p_k^0 \equiv \omega_i$. Then, since k_i is the momentum and ω_i is the energy of each of the pions j and k , the relation $\omega_i^2 = k_i^2 + m^2$ holds. The k^0 integration removes the i in Equation (III.4) and then the singularity in the $\vec{k}$ integral produces an imaginary part in I(i). Now, the square of the total τ matrix element is the quantity that one would eventually like to obtain, since only this quantity has physical meaning. The matrix element is the sum of the contributions from the basic diagram and all of the final-state interaction diagrams, and it has the general form

$$M_{\tau,s} \sim 1 + C\,I(1) + C\,I(2) + C\,I(3) \quad ,$$

where the C's are constants. To lowest order, the squares of the final-state interaction contributions can be dropped in $|M_{\tau,s}|^2$. Then it is easy to see that the imaginary parts of the I(i) cancel in the evaluation of $|M_{\tau,s}|^2$. Therefore, only the real part of I(i), given by the principal part of the integration, need be considered.

The principal part integration is logarithmically divergent, so that a subtraction must be made at some suitable point in order to remove the divergent term. The details of the calculation of I(i) are given in Appendix F and the result is

$$I(i) = \frac{1}{16\pi^2}\left(\frac{k_i}{\omega_i}\ln\frac{\omega_i - k_i}{m} - \frac{k_0}{\omega_0}\ln\frac{\omega_0 - k_0}{m}\right) .$$

The kinetic energy available to the pions is small ($\simeq$ 75 Mev), so that $k_i << \omega_i$. Then, to order k

$$\ln\frac{\omega_i - k_i}{m} \simeq -\frac{k_i}{m} \quad ,$$

and the expression for I(i) reduces to

$$I(i) \approx \frac{1}{16\pi^2 m}\left(\frac{k_0^2}{\omega_0} - \frac{k_i^2}{\omega_i}\right) . \qquad \text{(III.5)}$$

It is more useful to have the results in the rest frame of the kaon. The translation of all momenta to this frame is done as follows. In their centre of mass, the two interacting pions have energies ω_i and momenta $\vec{k}_i$ and $-\vec{k}_i$. If the third pion has momentum $\vec{p}_i$ with respect to the centre of mass of the kaon, then the centre of mass of the two interacting pions has momentum $-\vec{p}_i$. The effective "mass" of the two-pion system is $2\omega_i$, so that from conservation of energy one has

$$M = W_i + \sqrt{4\omega_i^2 + \vec{p}_i^{\,2}} , \qquad \text{(III.6)}$$

where M is the kaon mass and $W_i = \sqrt{p_i^2 + m^2}$ is the energy of the i^{th} (noninteracting) pion. The pions are non-relativistic, so that in good approximation

$$W_i = m + T_i ,$$

where T_i is the kinetic energy of the i^{th} pion. Using this approximation and the relation $\omega_i^2 = k_i^2 + m^2$, Equation (III.6) can be rewritten, after some manipulation, as

$$k_i^2 = \frac{M^2 - 2M(m + T_i) - 3m^2}{4} , \qquad \text{(III.7)}$$

which gives the relation between the momentum of the jth and kth pions and the kinetic energy of the ith pion.

The maximum value of T_i occurs for the minimum k_i^2 which is 0, so that

$$T_{max} = \frac{M^2 - 3m^2}{2M} - m \simeq 50 \text{ Mev}$$

and Equation (III.7) can be rewritten in a more simplified notation as

$$k_i^2 = \frac{MT_{max}}{2}\left(1 - \frac{T_i}{T_{max}}\right)$$

or

$$k_i^2 = m^2\rho^2(1 - t_i) , \qquad \text{(III.8)}$$

where t_i is the kinetic energy of the ith pion in units of T_{max}, and

$$\rho^2 = \frac{MT_{max}}{2m^2} \simeq 0.64 .$$

Defining an expression analogous to (III.8) for the subtraction term and using $\omega^2 = k^2 + m^2$, Equation (III.5) be-

comes

$$I(i) = \frac{1}{16\pi^2}\left[\frac{\rho^2(1 - t_0)}{\left(1 + \rho^2(1 - t_0)\right)^{\frac{1}{2}}} - \frac{\rho^2(1 - t_i)}{\left(1 + \rho^2(1 - t_i)\right)^{\frac{1}{2}}}\right]$$

$$= \frac{1}{16\pi^2}\, h(t_i) \; . \qquad \text{(III.9)}$$

Recall that the total matrix element is the sum of the basic diagram contribution and the final-state interaction diagram contributions. For τ-decay the basic interaction is given by 2g. Thus, substituting Equation (III.9) into Equation (III.3) and adding this correction term to the basic vertex, one finds the τ matrix element corrected by final-state s-wave pion-pion interactions to be

$$\frac{1}{2}\, M_{\tau,s} = g\left\{1 + \frac{2a_0}{\pi}\left(\frac{5}{3}h(t_1) + \frac{5}{3}h(t_2)\right)\right.$$

$$\left. + \frac{2a_2}{\pi}\left(\frac{1}{3}h(t_1) + \frac{1}{3}h(t_2) + 2h(t_3)\right)\right\} \; .\text{(III.10)}$$

If one chooses the subtraction point as the symmetric point t_0 defined by

$$3t_0 = t_1 + t_2 + t_3 = \frac{M - 3m}{T_{max}} \simeq 1.5 \quad ,$$

one observes that, since the functions $h(t_1)$ and $h(t_2)$ occur in the matrix element with the same coefficients, and

since these functions are linear in t, one can write $t_1 + t_2 = 3t_0 - t_3$, and thus eliminate t_1 and t_2 from the τ matrix element. Then, using the approximation of replacing t_i by $t_0 \simeq \frac{1}{2}$ in the denominator of $h(t_i)$, which involves an error of at most 15%$^{(15)}$, the matrix element assumes the simple form

$$\tfrac{1}{2}M_{\tau,s} \sim 1 - \frac{5}{6\pi} \frac{\rho^2}{(1 + \frac{1}{2}\rho^2)^{\frac{1}{2}}} (a_0 - a_2)(2t_3 - 1) \ , \qquad \text{(III.11)}$$

where the coupling constant, g, has been omitted.

It is seen that the correction term for the τ-decay matrix element depends, essentially linearly, on the kinetic energy of the unlike pion. The coefficient of $(2t_3 - 1)$ is determined by the difference of the I=0 and I=2 scattering lengths. Therefore, if the final-state pions scatter only in the s-wave, a study of the energy spectra of the pions will determine $a_0 - a_2$. Gell-Mann and Rosenfeld$^{(19)}$, using a plot compiled by Dalitz$^{(20)}$, have shown that a good fit to the τ-decay date is obtained by

$$M_\tau \sim 1 + 0.1(2t_3 - 1) \ . \qquad \text{(III.12)}$$

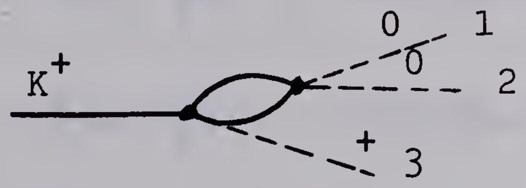
If only s-wave π-π interactions occur, this suggests that

$$a_2 - a_0 \simeq 0.7 \ .$$

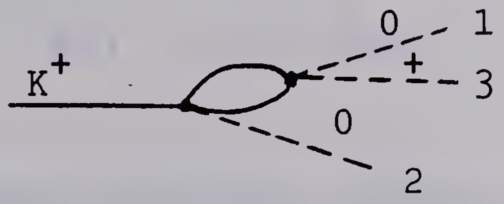
However, even though p-wave interactions are suppressed by a centrifugal barrier, they will still have some effect on the matrix element. These effects are discussed in Section III.3 .

III.2 τ' Decay of the K^+ Meson ($K^+ \to \pi^+\pi^0\pi^0$)

The analysis for τ' decay is the same as for decay. Here again, labelling the odd pion "3" and the other two "1" and "2", three types of diagrams contribute to the correction to the matrix element:

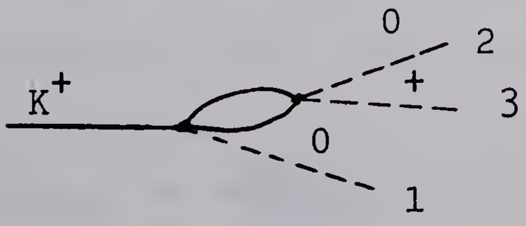


2(a)



2(b)

and



2(c)

Exactly as in τ-decay, the summation over intermediate states yields

$$2\times 2gI(3)T_{+-,00} \quad + \quad gI(3)T_{00,00}$$

for Diagram 2(a),

$$2gI(2)T_{+0,+0}$$

for Diagram 2(b), and

$$2gI(1)T_{+0,+0}$$

for Diagram 2(c). A Clebsch-Gordan analysis shows that

$$T_{+-,00} = \tfrac{1}{3}A_0 - \tfrac{1}{3}A_2 \simeq 32\pi(\tfrac{1}{3}a_0 - \tfrac{1}{3}a_2)$$

$$T_{00,00} = \tfrac{1}{3}A_0 + \tfrac{2}{3}A_2 \simeq 32\pi(\tfrac{1}{3}a_0 + \tfrac{2}{3}a_2)$$

and $$T_{+0,+0} = \tfrac{1}{2}A_2 \simeq 32\pi(\tfrac{1}{2}a_2) \quad .$$

Then, using the same h-functions defined previously, one finds that

$$M'_{\tau';s} = g\, a_0 \frac{10}{3\pi}\, h(t_3) + \frac{g a_2}{\pi}\left[2h(t_1) + 2h(t_2) - \tfrac{4}{3}h(t_3)\right] \tag{III.13}$$

Now, the basic τ' vertex has the strength g, so that, factoring out the g in the correction term (rather than 2g as in τ-decay) and adding the two terms, one can write the matrix element as

$$M_{\tau',s} \sim 1 + \frac{5}{3\pi} \frac{\rho^2}{(1 + \frac{1}{2}\rho^2)^{\frac{1}{2}}}(a_0 - a_2)(2t_3 - 1) \quad . \tag{III.14}$$

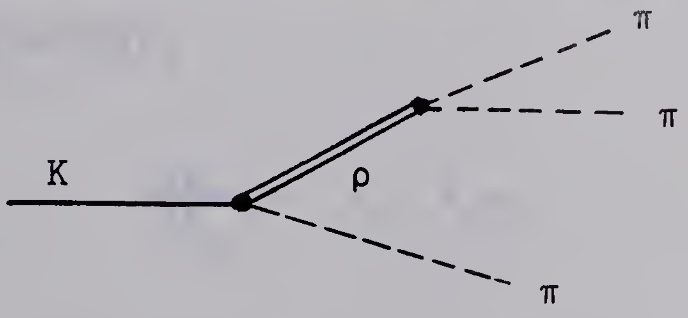
One sees that the correction term in this matrix element is also linear in t_3, just as was found for the τ matrix element. For τ' decay however, the coefficient of t_3 has twice the magnitude and the opposite sign.

These results for the effect of final-state interactions agree with the results obtained by Khuri and Treiman(15). If the pions were emitted in strictly an s-state, then the value of $a_0 - a_2$ could be determined by a Dalitz analysis of K^+ decay data. However, the two interacting pions can be in a p-state of angular momentum with the third pion also in a p-state with respect to the kaon rest frame so as to make the total angular momentum zero. The lowest-mass two-pion p-state resonance known is the ρ meson. This resonance dominates π-π p-wave scattering at energies of about 750 Mev. It will be assumed that the resonance also determines low-energy π-π scattering. Then, the sum of the basic vertex plus the s-wave final-state interaction corrections plus the p-wave final-state interaction corrections is expected to describe the actual kaon decay within a small error, since higher angular momenta

are suppressed by a centrifugal barrier.

III.3 p-wave π-π Interactions in τ and τ' Decays

With the assumption of ρ-dominance of the p-wave π-π interaction, the diagram corresponding to $K \rightarrow 3\pi$ decay is



In attempting to write a suitable Lagrangian for this diagram, it is instructive to first consider the process $\rho \rightarrow 2\pi$, which is exhibited in the right-hand vertex in the above diagram.

The rho meson is a vector meson(21), and, since it must obey overall Bose-Einstein statistics, has isospin 1. The rho decays only into two pions, which must be emitted into an ℓ=1 state by conservation of angular momentum. This state has odd parity and hence, since the strong interaction conserves parity, the ρ also has odd parity. Therefore, a suitable Lagrangian for the rho decaying into two pions is(22)

$$L = i\, f_s\, \vec{\rho}_\mu \cdot \left(\vec{\phi} \times \frac{\partial \vec{\phi}}{\partial x_\mu} \right) \quad , \qquad \text{(III.15)}$$

where the vector signs indicate isovectors. Notice that the pions are in an antisymmetric state, as required. The decay of the ρ_0 into two neutral pions, for example, is not allowed, since the two identical bosons would have to be in a symmetric state, implying an even value for ℓ.

The four-derivative operating on the pion field will produce a factor proportional to the pion's four-momentum. Explicitly,

$$\frac{\partial \phi}{\partial x_\mu} = \pm\, i\, k_\mu \phi \quad , \qquad \text{(III.16)}$$

where the plus sign applies if the field annihilates a pion and the minus sign applies if the field creates a pion.

The Lagrangian can be written in another form as

$$L_s = \frac{i}{2} f_s\, \varepsilon_{rst}\, \rho_{\mu r} \left(\phi_s \frac{\overleftrightarrow{\partial}}{\partial x_\mu} \phi_t \right) \quad , \qquad \text{(III.17)}$$

where ε_{rst} is the antisymmetric tensor with rst the isovector indices of the fields, and $\overleftrightarrow{\partial}$ is defined by

$$A \frac{\overleftrightarrow{\partial}}{\partial x_\mu} B \equiv A \frac{\partial B}{\partial x_\mu} - \frac{\partial A}{\partial x_\mu} B \quad .$$

The factor ½ in (III.17) is included because each term in the expansion of L occurs twice - once from the sum over r,s,t and once from the operator $\overleftrightarrow{\partial}$. Note that the Lagran-

gian (III.15) is formally identical with (III.17) only because the isovector product involves two identical fields. If the fields are different, the form (III.17) must be used.

In the usual way, the components of the isovector fields may be written in terms of the charged fields as (See introduction to this chapter)

$$\begin{aligned} \phi_1 &= \frac{\phi_+ + \phi_-}{\sqrt{2}} \\ \phi_2 &= \frac{\phi_+ - \phi_-}{i\sqrt{2}} \qquad\qquad (III.18) \\ \phi_3 &= \phi_0 \quad . \end{aligned}$$

The same relations hold for the ρ-meson field. Then, using the relation (III.16), the r,s,t = 1,2,3 term, for example, in the expansion of (III.17) is

$$\frac{(\rho_+ + \rho_-)_\mu}{2i}\Big(\phi_+(-ik_0)_\mu\phi_0 - \phi_-(-ik_0)_\mu\phi_0 - (-ik_+)_\mu\phi_+\phi_0 + (-ik_-)_\mu\phi_-\phi_0\Big)$$

which reduces to

$$\frac{(\rho_+ + \rho_-)_\mu}{2}\Big((k_+ - k_0)_\mu\phi_+\phi_0 + (k_0 - k_-)_\mu\phi_-\phi_0\Big) \quad .$$

The r,s,t = 1,3,2 term is the negative of this with the charge indices in the expression reversed. Since this also reverses the sign of the momentum term, the fields and momenta are seen to occur in the combination, for example,

$$(k_+ - k_0)(\phi_+\phi_0 + \phi_0\phi_+) \quad .$$

Now, in the process relevant to $K^+ \rightarrow 3\pi$ decay, either ρ^+ or ρ^0 is annihilated to produce positive and neutral pions. (A ρ^- can never be involved in $K^+ \rightarrow 3\pi$ decays because it decays into $\pi^- + \pi^0$.) Then, expanding the rest of the Lagrangian in a similar way and retaining only terms pertinent to $K^+ \rightarrow 3\pi$ decay, one finds

$$L_s = f_s \varepsilon_\mu \Big\{ (k_+ - k_0)_\mu \rho_- (\phi_+\phi_0 + \phi_0\phi_+) + (k_- - k_+)_\mu \rho_0 (\phi_+\phi_- + \phi_-\phi_+) \Big\} , \tag{III.19}$$

where ε_μ is the polarization vector of the ρ-meson.

One sees from the above expression that the ρ-meson decays into antisymmetric two-pion states and that the effective coupling is the same ($\sim f_s$) for both the charged and uncharged ρ decay. The charged ρ decay appears in τ' kaon decay and the uncharged ρ decay appears in τ kaon decay.

Now, the K-ρ-π vertex must be considered. First of all, if the $|\Delta I| = \frac{1}{2}$ rule is valid, the singly-charged final state must have isospin 1 with $I_3 = 1$. The $(I;I_3) = (1;1)$ isospin wave function in terms of the rho and pion fields is

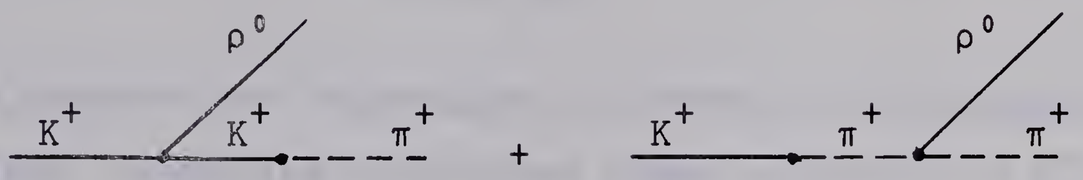
$$|1;1\rangle = \frac{1}{\sqrt{2}}\Big(|\phi_+\rho_0\rangle - |\rho_0\phi_+\rangle\Big) ,$$

which is antisymmetric under the exchange of the charge indices of the two particles. Also, a consideration of the "crossed" reaction $\rho \to K + \pi$ shows that the kaon and pion must be in a relative p-state of angular momentum. Consider the four-vector $(k - K)$, where k is the pion four-momentum and K is the kaon four-momentum. In the centre of mass of the two particles, one has

$$(k - K) = \{(\omega_\pi - \omega_K) , 2\vec{k}\} . \tag{III.20}$$

If the kaon and pion masses are equal, this expression is antisymmetric under exchange of the K and π, so that in a higher symmetry where $m = M$, the pure K-π p-state is represented by a term of the form $(k - K)$. Therefore, one can make the approximation of representing the K-π p-state with $m \neq M$ by the term (III.20). This approximation is further

justified by the fact that, if one expands the K-ρ-π vertex into the components



a term such as (k - K) is automatically produced. (Actually, the momentum term occurs explicitly as (k + K), which is the antisymmetric term for one particle in the initial state and one particle in the final state.)

In the above approximation, the phenomenological Lagrangian for the K-ρ-π vertex has the same form as that for the ρ-π-π vertex. That is,

$$L_w = \frac{i}{2} f_w \varepsilon_{rst} \rho_{\mu r} \left(\Phi_s \frac{\overleftrightarrow{\partial}}{\partial x_\mu} \phi_t \right) , \qquad \text{(III.21)}$$

where $\vec{\Phi}$ is the fictitious spurion-kaon isovector field, $\overline{S}\,\vec{\tau}\,K$. The subscript w on the coupling constant f_w indicates that this is a weak vertex, whereas the ρ-π-π vertex, with coupling constant f_s, is strong. The isovector components of $\vec{\Phi}$ are given by

$$\Phi_1 = (S_1^* \; S_2^*) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = S_1^* K^0 + S_2^* K^+$$

$$\Phi_2 = (S_1^* \; S_2^*) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} K^+ \\ K^0 \end{pmatrix} = -iS_1^* K^0 + iS_2^* K^+$$

$$\Phi_3 = (S^* \; S^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}K^+\\ K^0\end{pmatrix} = S^*K^+ - S^*K^0 \qquad \text{(III.22)}$$

One now expands the Lagrangian in terms of the charged meson fields, retaining only those terms relevant to $K^+ \to 3\pi$ decay and omitting the spurion fields, and obtains

$$L_W = \sqrt{2}\, f_W\, \varepsilon_\mu\, (k + K)_\mu \left(K^+\rho^+\pi^0 - K^+\rho^0\pi^+\right) \qquad \text{(III.23)}$$

The factor $\sqrt{2}$ arises because the fields Φ_1, Φ_2, Φ_3 have not been normalized. This factor will be absorbed into the coupling constant f_W. Notice that if the $K^+ \to \rho^0\pi^+$ vertex (τ-decay) is characterized by the coupling constant f_W, then the $K^+ \to \rho^+\pi^0$ vertex (τ'-decay) is characterized by $-f_W$.

Having established the isospin properties of the two vertices in the charged kaon decay, one now must include the dynamics of the process arising from the presence of the virtual ρ-meson. τ-decay involves the decay of a ρ^0 into $\pi^+\pi^-$ and τ' decay involves the decay of a ρ^+ into $\pi^+\pi^0$. Hence, the pion of unlike charge, the "odd" pion, always takes part in the final-state interaction. As was done before, the odd pion is labelled by "3" and the other two pions are labelled by "1" and "2". If I(i) is defined as the contribution to the matrix element from the process

in which pions j and 3 interact via the ρ channel, then

$$M'_{\tau,p} = I_\tau(1) + I_\tau(2) \qquad \text{(III.24)}$$

and
$$M_{\tau',p} = I_{\tau'}(1) + I_{\tau'}(2) \; . \qquad \text{(III.25)}$$

From the form of the Lagrangians it is evident that $I_\tau(i) = -I_{\tau'}(i)$, so that only $I_\tau(i)$ need be calculated. To calculate $I_\tau(i)$, one combines the two Lagrangians and takes the matrix element between the initial and final states for τ-decay. Then, for $i,j = 1$ or 2,

$$I_\tau(i) = f_w f_s \int d^4k \, (k_i + K)_\mu \frac{\delta_{\mu\nu} - k_\mu k_\nu / m_\rho^2}{k^2 - m_\rho^2} (k_3 - k_j)_\nu$$
$$\times \; \delta^4(K - k_i - k)\,\delta^4(k - k_j - k_3) , \qquad \text{(III.26)}$$

where the factor in the numerator of the ρ-meson propagator comes from the sum over the polarizations of the ρ(23). Here K represents the four-momentum of the kaon and k_i, k_j, and k_3 represent the four-momenta of the pions. As in the case of s-wave final-state interactions, the normalization factors common to both the basic vertex and the correction term have been omitted. Evaluating the integral and leaving out the overall energy-momentum conserving δ-function, one has

$$I_\tau(i) = \frac{f_s f_w}{(K - k_1)^2 - m_\rho^2} \left\{ (K + k_i)\cdot(k_3 - k_j) - \frac{1}{m_\rho^2} \left[(K + k_i)\cdot(K - k_i) \right] \left[(K - k_i)\cdot(k_3 - k_j) \right] \right\} . \tag{III.27}$$

With the assumption that the pion masses are degenerate, the last term gives no contribution, since, by energy-momentum conservation, $K - k_i = k_j + k_3$ and $k_3^2 - k_j^2 = m_{\pi_3}^2 - m_{\pi_j}^2 = 0$.

At this stage it is convenient to define the three scalar variables:

$$s_i = (K - k_i)^2 \qquad i = 1,2,3 \quad .$$

Then, using $K + k_i = 2K - k_j - k_3$, one finds, for pions 2 and 3 interacting

$$I_\tau(1) = \frac{f_s f_w}{-s_1 + m_\rho^2} (s_2 - s_3) \ . \tag{III.28}$$

$I_\tau(2)$ is the same as $I_\tau(1)$ with the indices 1 and 2 interchanged. Then, the p-wave correction to the τ matrix element is (Compare Riazuddin and Fayyazuddin - 1961)$^{(24)}$

$$M'_{\tau,p} = f_s f_w \left(\frac{s_2 - s_3}{-s_1 + m_\rho^2} + \frac{s_1 - s_3}{-s_2 + m_\rho^2} \right) . \tag{III.29}$$

This result for the p-wave correction may be translated to the rest frame of the kaon. First, one notes that the maximum value for s_1 or s_2 is $(M - m)^2 \simeq 6.25\ m^2$ and that $m_\rho^2 \simeq 29\ m^2$. Therefore, it is a reasonable approximation to neglect s_1 and s_2 in the denominators of the correction terms. Then, since

$$s_1 + s_2 + s_3 = (K-k_1)^2 + (K-k_2)^2 + (K-k_3)^2 = M^2 + 3m^2 , \tag{III.30}$$

the contribution from the p-wave pion-pion interaction may be written as a function of s_3 only. That is,

$$M'_{\tau,p} = f_s f_w \left(\frac{M^2 + 3m^2 - 3s_3}{m_\rho^2} \right) . \tag{III.31}$$

In the centre of mass of the kaon, $K = (M , 0)$ and $k_3 = (\omega_3 , k_3)$, where $\omega_3^2 = k_3^2 + m^2$. Then, since the pions are non-relativistic, one may make the approximation

$$\omega_3 \simeq m + T_3 ,$$

where T_3 is the kinetic energy of the odd pion. Upon making these substitutions, one has

$$s_3 = (M - m)^2 - 2MT_3 . \tag{III.32}$$

As in the case of the s-wave final-state interactions, one defines the quantity

$$t_3 = T_3/T_{max} ,$$

where T_{max} is defined in Section III.1 and is approximately equal to 50 Mev. One also defines

$$\rho^2 = \frac{MT_{max}}{2m^2} ,$$

and, substituting these quantities into Equation (III.31), the p-wave correction to the τ matrix element can be simply written as

$$M'_{\tau,p} = f_s f_w \left(\frac{6m^2\rho^2(2t_3 - 1)}{m_\rho^2} \right) . \quad \text{(III.33)}$$

Now, recall that the strong ρ-π-π vertex has coupling constant f_s for both τ and τ' decays, but the weak K-ρ-π vertex has coupling constant f_w for τ decays and $-f_w$ for τ' decays. Therefore, the p-wave correction to τ' decay is just the negative of the correction to τ decay. That is,

$$M'_{\tau',p} = - f_s f_w \left(\frac{6m^2\rho^2(2t_3 - 1)}{m_\rho^2} \right) . \quad \text{(III.34)}$$

Notice that these p-wave terms have the same energy depen-

dence as the s-wave correction terms. One can now add Equations (III.33) and (III.34) to Equations (III.11) and (III.14) to obtain the total corrected τ and τ' matrix elements. Then, squaring and retaining terms only up to first order in the corrections, M_τ' and $M_{\tau'}'$, one finally obtains for the squares of the total matrix elements the results

$$\tfrac{1}{4}|M_\tau|^2 \sim 1 + \left[-\frac{5}{3\pi} \frac{\rho^2(a_0 - a_2)}{(1 + \frac{1}{2}\rho^2)^{\frac{1}{2}}} + \frac{6m^2\rho^2 f_s f_w}{g\, m_\rho^2} \right](2t_3 - 1) \qquad \text{(III.35)}$$

and

$$|M_{\tau'}|^2 \sim 1 - 2\left[-\frac{5}{3\pi} \frac{\rho^2(a_0 - a_2)}{(1 + \frac{1}{2}\rho^2)^{\frac{1}{2}}} + \frac{6m^2\rho^2 f_s f_w}{g\, m_\rho^2} \right](2t_3 - 1)\ . \qquad \text{(III.36)}$$

The factor g (the coupling constant for the basic or "no final-state interaction" K → 3π vertex) in the denominator of the p-wave term arises because a factor g has been factored out and then omitted from the squared matrix elements. The basic interaction term and the s-wave correction term are proportional to g, but the p-wave term is not; hence, the factor 1/g in the p-wave term.

Thus, the inclusion of p-wave pion-pion final-state interactions still leaves the energy-dependent parts

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of the squares of the τ and τ' matrix elements proportional to $(2t_3 - 1)$, where t_3 is the kinetic energy of the pion of unlike charge. Also, the coefficient of the kinetic energy term in the τ' matrix element remains twice as large as and opposite in sign to the coeffient of the kinetic energy term in the τ matrix element, just as was found when only s-wave π-π interactions were considered.

Now, the magnitude of the ρ-π-π coupling constant can be estimated from the mass and width of the ρ resonance. $M_\rho \sim 750$ Mev and $\Gamma_\rho \sim 100$ Mev, so that$^{(25)}$

$$f_s \sim \sqrt{48\pi\Gamma_\rho/M_\rho} \sim 4 \quad .$$

Hence, if one has a value for either one of $(a_0 - a_2)$ or f_w, the other can be readily calculated by using the experimental results for $K^+ \to 3\pi$ decay. Equation (III.12) gives the experimental value for $|M_\tau|^2$:

$$|M_\tau|^2 \sim 1 + 0.2(2t_3 - 1) \quad .$$

One can use, for example, Weinberg's result$^{(26)}$, which suggests that $a_0 = 0.2$ (which, incidentally, can be explained by any of the form factors proposed in Chapter II), and $a_2 = -0.06$. Then, $a_0 - a_2 = 0.26$, and this value can be substituted into Equation (III.35), together with the a-

bove value for f_s, and the ratio f_w/g can be calculated. This turns out to be

$$\frac{f_w}{g} \sim 0.5 \quad . \qquad (III.37)$$

Thus, if the (weak) K-ρ-π vertex has about one-half the strength of the (weak) K-3π vertex, a difference of 0.26 between the I=0 and I=2 s-wave π-π scattering lengths will lead to the correct τ matrix element. If the ratio f_w/g is less than about 0.4 it will take a negative value of $(a_0 - a_2)$ to reproduce experimental results, and if f_w/g is zero $(a_0 - a_2)$ turns out to be approximately -0.7. It would be very useful, then, to have some accurate estimate of f_w, since g can be calculated from the τ decay rate of K^+ and hence the difference $(a_0 - a_2)$ could be determined with, one believes, a fair degree of accuracy.

CHAPTER IV $K_{\ell 4}$ DECAY

In Chaper II, various parametrized forms for the K-π-π form factor were proposed, and the parameters were adjusted to fit experimental data, in this case the $K_1^0 - K_2^0$ mass difference. Inasmuch as the energy dependence was fixed by the type of parametrization used, one could not obtain explicit information about the structure of the form factor; one could only say that, if the form factor has some particular parametrized structure, then the parameters have certain possible values. In this chapter, a slightly different approach will be used. The hadronic (K-π-π) part of $K_{\ell 4}$ decay will be written in its most general form and the matrix element will be calculated assuming that all the energy dependence in the process comes from pion-pion final-state interactions. Then the energy dependent part will be identified with certain K-π-π form factors.

If one assumes a current-current type interaction for the $K_{\ell 4}$ process, the Hamiltonian is written

$$H = J_\mu j_\mu ,$$

where J_μ is the hadron current and j_μ is the leptonic current. Then the matrix element for this process is

$$M = \sum_n <2\pi\ell\bar{\nu}|J_\mu|n><n|j_\mu|K>$$

where the sum is over all possible intermediate states. Since J_μ contains only hadron field operators and j_μ contains only lepton field operators the matrix element can be factored as

$$M = <2\pi|J_\mu|K><\ell\nu|j_\mu|o> \quad .$$

The right-hand part is the well-known leptonic matrix element, which provides information about the behaviour of the leptons. It will not be considered further.

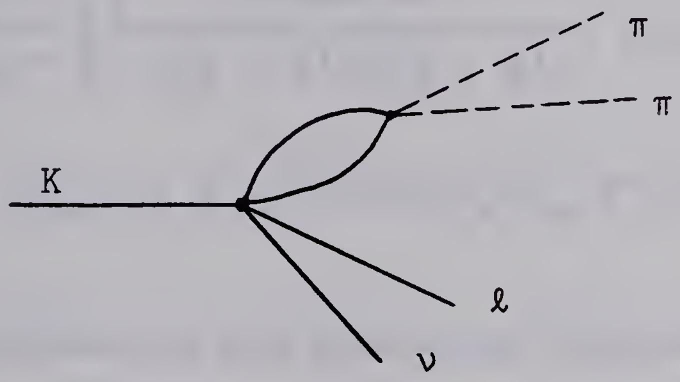
The hadronic matrix element can be written in its most general form as(27)

$$A_\mu = F_1(p_1 + p_2)_\mu + F_2(p_1 - p_2)_\mu$$

$$+ F_3(K - p_1 - p_2)_\mu + F_4\ \varepsilon_{\mu\nu\rho\sigma}K_\nu p_{1\rho} p_{2\sigma} \qquad \text{(IV.1)}$$

where K, p_1, p_2 are the exterior four-momenta of the kaon and the two pions respectively. A_μ is the matrix element for the point interaction. In this most general form the

form factors are functions of the invariants K^2, p_1^2, p_2^2, $K\cdot p_1$, $K\cdot p_2$ and $p_1\cdot p_2$. However, initially it will be assumed that the F_i's are constants for the basic point interaction and that all the energy dependence of the matrix element comes from a process represented by the diagram



The pions may scatter in the symmetric s-state with I = 0 or 2, or in the antisymmetric p-state with I = 1. The term $F_1(p_1 + p_2)_\mu$ in the matrix element is symmetric in p_1 and p_2 and therefore represents s-wave scattering. Similarly, $F_2(p_1 - p_2)_\mu$ represents p-wave scattering.

IV.1 s-wave Pion-Pion Interactions

In the expression for A_μ, the F_2 and F_4 terms are antisymmetric in p_1 and p_2 and therefore will not contribute to the s-wave process. The F_3 term contains the factor $(K - p_1 - p_2)_\mu$, which, by energy-momentum conservation, is equal to the four-momentum of the leptons. Therefore,

when it couples with the leptonic part of the matrix element, it will produce terms of order m_ℓ, where m_ℓ is the mass of a lepton, and so it will be neglected. Then, the matrix element for the s-wave π-π interaction is

$$M_\mu^h = \frac{i^2}{2(2\pi)^4} \iint \frac{d^4k_1 d^4k_2}{(k_1^2 - m^2)(k_2^2 - m^2)} F_1(k_1 + k_1)_\mu$$

$$\times \; iT_{\alpha\beta,\gamma\delta} \; \delta^4(K-k_1-k_2-k_\ell-k_\nu)\delta^4(k_1+k_2-p_1-p_2) \;, \tag{IV.2}$$

where $iT_{\alpha\beta,\gamma\delta}$ represents the pion-pion scattering process. One integration will remove one of the δ-functions and the other δ-function will be absorbed into the normalization. The remaining integral is identical with Equation (III.4) and the calculation is done in Appendix F. In this case, however, the imaginary part will not be neglected. The integral is logarithmically divergent so that a subtraction was performed to remove the divergence. Then, choosing the subtraction point at threshold ($k_0 = 0$), one has

$$M_\mu^h = F_1(p_1 + p_2)_\mu T_{\alpha\beta,\gamma\delta} \left(\frac{k}{16\pi^2\omega} \ln\frac{k + \omega}{m} + \frac{ik}{32\pi\omega} \right) , \tag{IV.3}$$

where the matrix element is given in the centre of mass of the two pions.

If one assumes the validity of the $\Delta S = \Delta Q$ rule, then, in the decay of a charged kaon into two pions and a

lepton pair, the leptons carry the charge of the kaon and the pion pair is neutral. The two possible neutral pion pairs are (+-) and (00). In this paper, only the decay $K^+ \to \pi^+ + \pi^- + \ell + \overline{\nu}$ will be considered. The forms of T applicable to this process can be found by the technique used in Chapter III. These are

$$T_{+-,+-} \simeq 32\pi\left(\frac{1}{3}a_0 + \frac{1}{6}a_2\right) \tag{IV.4}$$

and

$$T_{00,+-} \simeq 32\pi\left(\frac{1}{3}a_0 - \frac{1}{3}a_2\right),$$

where a_0 and a_2 are the I=0 and I=2 s-wave scattering lengths. The summation over intermediate states includes (+-), (-+), and (00). The dynamics are the same for each of these cases so that one simply adds the three T-matrices. Notice that the I=2 scattering length will not contribute. Then, adding the expression (IV.3) to the contribution from the basic vertex, one has for the total hadronic matrix element

$$M^h_\mu = F_1(p_1 + p_2)_\mu\left[1 + \frac{2a_0k}{\pi\omega}\ln\frac{k+\omega}{m} + \frac{ia_0k}{\omega}\right]. \tag{IV.5}$$

By time reversal invariance$^{(28)}$, the s-wave matrix element can be written in the form

$$M_\mu^h = F_1(k^2)\, e^{i\delta_0}\, (p_1 + p_2)_\mu \,, \qquad \text{(IV.6)}$$

where δ_0 is the s-wave phase shift and $F_1(k^2)$ is real. To write Equation (IV.5) in the form (IV.6), one sees that one must have

$$F_1(k^2) = F_1 \left\{ \left[1 + \frac{2a_0k}{\pi\omega} \ln\frac{k+\omega}{m} \right]^2 + \frac{a_0^2k^2}{\omega^2} \right\}^{\frac{1}{2}} . \qquad \text{(IV.7)}$$

Also, if the phase shift is small, one can write

$$e^{i\delta_0} \simeq 1 + i\delta_0 \,,$$

so that at low energies ($k << \omega$) one can make the identification

$$\delta_0 \approx \arctan \left(\frac{a_0k/\omega}{1 + \frac{2a_0k}{\pi\omega} \ln\frac{k+\omega}{m}} \right) . \qquad \text{(IV.8)}$$

One can observe that this phase shift is identical with the Chew - Mandelstam(29) parametrization for the phase shift.

The energy-dependent form factor (IV.7) is seen to diverge logarithmically at high energies. This is contrary to the expected general behaviour of the form fac-

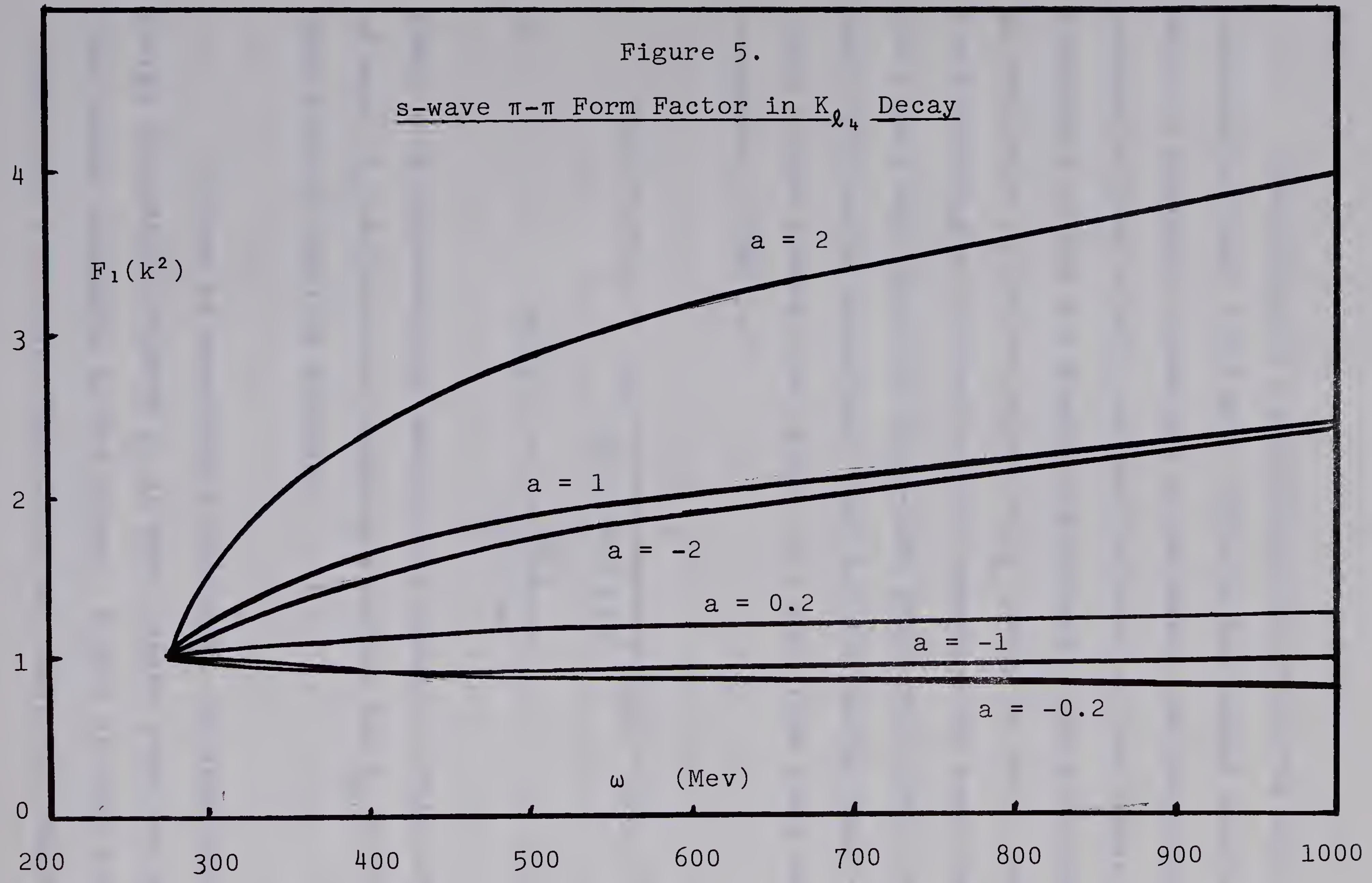
tors. However, this expression was derived assuming that the pion-pion T-matrix is constant and while one hopes that this approximation is valid in the low-energy region, one certainly does not have as much confidence in the approximation at higher energies. In any case, in the semileptonic decay of the kaon at rest, there is an upper limit to the possible energies of the pions because of energy conservation, and so there is an upper limit to the range in which the form factors can be experimentally determined. This means that the extreme high-energy behaviour of the form factor is irrelevant to the problem of kaon decays. The energy dependence of the form factors is displayed in Figure 5 for various values of a_0.

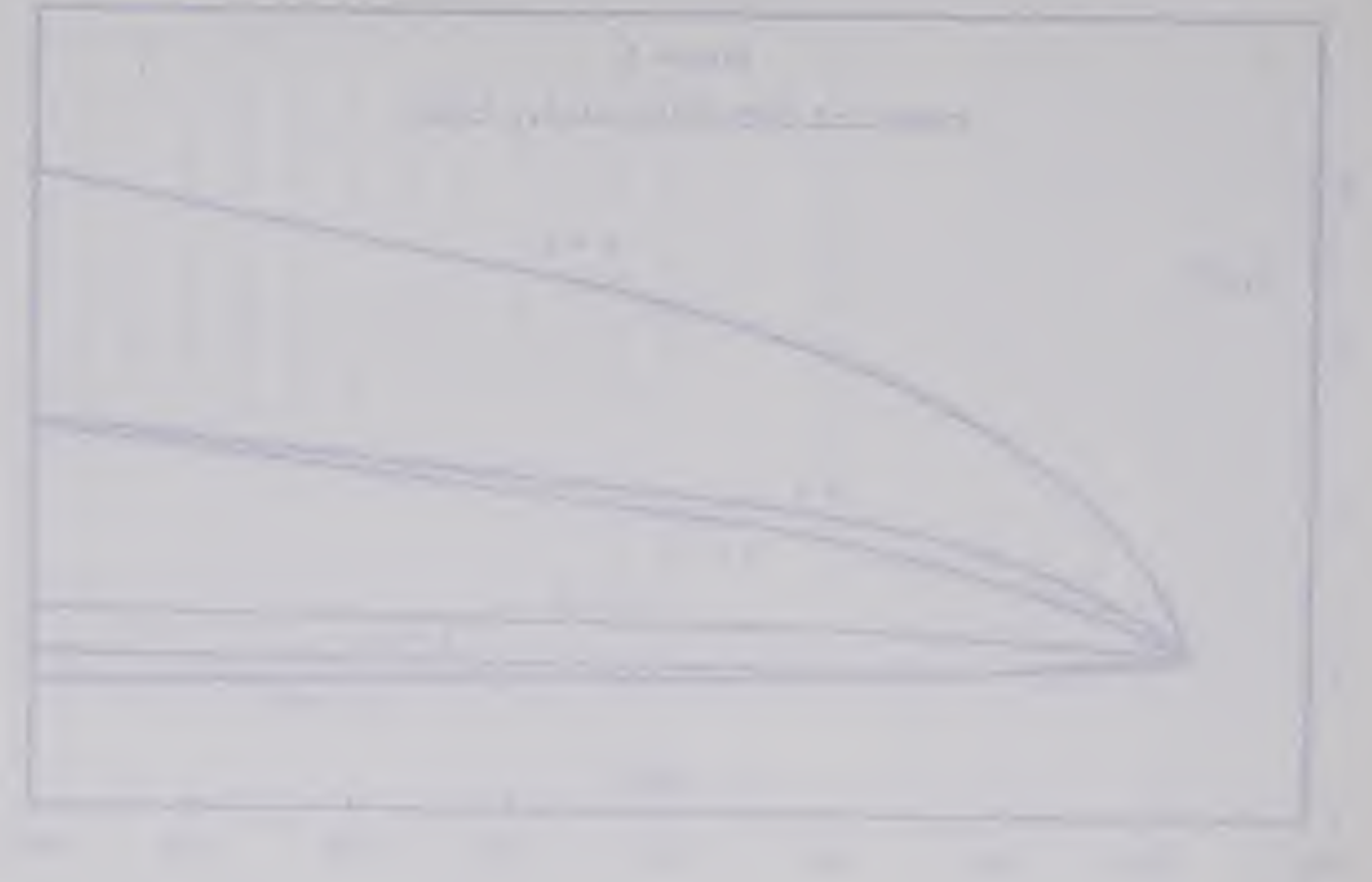
IV.2 p-wave Pion-Pion Interaction

The term in (IV.1) that contributes to the process containing p-wave π-π interactions is $F_2(p_1 - p_2)_\mu$. The F_4 term can also make some contribution. However, notice that, in the centre of mass of the kaon, $K = (\omega_K, 0)$ so that $\vec{p}_2 = -\vec{p}_1 - (\vec{p}_\ell + \vec{p}_\nu)$. Then, since $\varepsilon_{\mu\nu\rho\sigma}K_\nu q_\rho q_\sigma = 0$ from the definition of ε, the F_4 term reduces to $F_4 \varepsilon_{\mu 0 \rho\sigma} K_0 p_{1\rho} (p_\ell + p_\nu)_\sigma$, which produces terms of the order of the lepton mass. Therefore, the F_4 term can be neglec-

Figure 5.

s-wave π-π Form Factor in K_{ℓ_4} Decay





ted.

Two pions in a p-state can interact via the ρ resonance of mass ≃ 770 Mev. This is the lowest mass p-wave π-π resonance known and so one makes the reasonable assumption that it will account for most of the energy dependence of the π-π scattering process. Since the energy available to the two pions in K_{ℓ_4} decay is small, the π-π scattering will take place at energies far away from the ρ mass and therefore the p-wave phase shift will be small and can be neglected. Then all the energy dependence of the p-wave form factor will come from the ρ-meson propagator. That is

$$F_2(p_1 - p_2)_\mu = g_w \frac{1}{m_\rho^2 - (p_1 + p_2)^2} f_s(p_1 - p_2)_\mu \quad \text{(IV.9)}$$

or

$$F_2(k^2) = \frac{g_w f_s}{m_\rho^2 - 4\omega^2} , \quad \text{(IV.10)}$$

where ω is the energy of each of the pions in their centre of mass, f_s is the ρ-π-π coupling constant and g_w is the weak K-ρ-ℓ-$\bar{\nu}$ coupling constant.

It can be seen that $F_2(k^2)$ will not show much energy dependence because m_ρ is much larger than the maximum energy available to the pions. $F_1(k^2)$ is also slowly varying, but its energy dependence has different character-

istics for opposite signs of a_0, the I=0 s-wave scattering length. For $a_0>0$, the form factors have positive slope everywhere and at high energies they increase as $(\ln k)^2$. For $a_0<0$, on the other hand, the energy dependence is more complex. By differentiating (IV.7) twice, one finds that for $\frac{-4}{\pi} < a_0 < 0$, the form factor decreases initially as k rises from 0. Then, as k increases further, the form factor increases until, at very high energies, it approaches $(\ln k)^2$. For $a_0 < \frac{-4}{\pi}$, the form factor has positive slope everywhere, just as it had for $a_0 > 0$.

The $K^+ \rightarrow \pi^+ + \pi^- + \ell + \bar{\nu}$ decay is useful for the study of pion-pion interactions because the hadronic matrix element contains only a_0, and not $(a_0 - a_2)$ as in the $K^+ \rightarrow 3\pi$ decay. Therefore, if the K-ρ-ℓ-$\bar{\nu}$ coupling constant, g_W, were known, the I=0 s-wave π-π scattering length could be determined for the simple models used here. Then a_0 could be obtained using the method outlined in Chapter III.

An analysis of experimental data has been done by Berends et al(30) using form factors of the type derived here. The results are not altogether satisfactory, so that more work must be done on this problem.

CHAPTER V CONCLUSION

The analysis of kaon-pion processes done in this work has led to several observations concerning the character of the pion-pion scattering process.

First of all, in Chapter II it was found that the π-π phase shifts exhibit the most rapid variation at the points where the form factor is largest. For example, in the case of narrow resonances, the phase shifts rise rapidly through 90° at the resonance mass. However, the value of $|F(k^2)|^2$ at the resonance mass is k_r^2/γ^2 so that if the resonance is very broad, k_r^2/γ^2 could be less than 1 and hence, the maximum value of $|F(k^2)|^2$ will be at $k^2 = 0$. In this case, the phase shift will have the steepest slope at zero π-π relative momentum. Similarly, since the first parametrization and the effective range parametrization form factors are monotonically decreasing functions of the π-π energy (for a and r_0 of the same sign in the effective range parametrization), the phase shifts vary most rapidly at $k^2 = 0$. This behaviour is to be expected because the form factor is, in a sense, a measure of the strength of the interaction and, at energies where the pions interact

most strongly, one expects the scattering to have the most structure.

It was also noticed that the phase shifts exhibit more energy dependence for more rapidly-varying form factors. This means that, for the first and third parametrizations, the scattering length is larger for more convergent form factors. However, the highly convergent form factors lead to unrealistic high-energy behaviour for the phase shifts. The fourth parametrization gives a good illustration of this fact. The form factor converges exponentially and produces phase shifts which actually diverge in the high-energy limit. The first parametrization phase shifts also behave unrealistically for quickly-converging form factors. Higher values of n increase the rate of convergence of the form factor and the corresponding phase shifts tend to $n\pi$ at high energies. The effective range parametrization phase shifts have perhaps the most realistic high-energy behaviour. The phase shifts rise to a maximum (or decrease to a minimum for $a < 0$, $r_0 < 0$) and then slowly go to zero at high energies. This is the behaviour expected of a system which has no bound state or resonance. If, however, an s-wave π-π resonance does exist, the phase shifts will be expected to go to 180° at high energies.

The first parametrization can lead to scattering lengths between 0.2 for $n = \frac{1}{2}$ and 0.31 for $n = 2$, and the

corresponding range of the phase shifts at the kaon mass is 17° to 27°. The second parametrization can lead to scattering lengths between 0.04 for $M_r = 510$ Mev, $\Gamma = 24$ Mev, and 0.25 for $M_r = 700$ Mev, $\Gamma = 357$ Mev; and the corresponding $\delta(M^2)$ are between 44° and 34°. This parametrization can lead to phase shifts of up to 90° in the limit of $\Gamma \to 0$ and M_r approaching the kaon mass (No resonances below the kaon mass are possible since this would lead to the wrong sign for ΔM.). The third parametrization, for a and r_0 of the same sign, leads to scattering lengths between -0.2 and 0.2 and $\delta(M^2)$ between -17° and 17°.

From these results one sees that it would require extreme values of the parameters to obtain scattering lengths above about 0.4 or $\delta(M^2)$ above 45°. If any of the form factors given provide a realistic description of the π-π scattering process, this would suggest that both a and $\delta(M^2)$ are, in fact, small. Also, $a > 0$ is preferred because, with $a < 0$, one also has $r_0 < 0$, which is unrealistic for simple scattering processes.

The consideration of $K \to 3\pi$ decay has shown that, if the pions in the final state only interact in s-waves, the difference $a_0 - a_2$ is approximately -0.7. Unless a_2 is of the order of $+1$, this result is not compatible with the results of Chapter II. However, if the pions can also

scatter in p-waves, it has been shown that a K-ρ-π coupling constant of the order of the basic K-3π coupling constant will lead to a value for $a_0 - a_2$ which varies considerably from -0.7. In particular, for $f_W \sim \frac{1}{2}g$, then $a_0 - a_2 \sim 0.25$. This result agrees with the current-algebra result$^{(26)}$, which suggests that $a_0 \simeq 0.2$ and $a_2 \simeq -0.06$, and is also more compatible with the results of Chapter II.

The form factor derived for K_{ℓ_4} decays is logarithmically divergent at high energies and leads to a phase shift which, if it retains its low-energy form in the high-energy region, tends to zero in the high-energy limit for positive scattering lengths and tends to -180° for negative scattering lengths. This is consistent with the general relations between the phase shift and form factor found in Chapter II. However, the phase shift was derived in the low-energy region, so that one cannot say much about its true behaviour at high energies.

With the experimental evidence on the K-meson processes considered in this work still incomplete, the conclusions reached are necessarily rather indefinite. When more experimental results become available, however, the methods described here can be used to determine the π-π phase shifts and scattering lengths more precisely,

and one will then have a much clearer understanding of the pion-pion scattering process.

APPENDIX A

Calculation of Re $\Sigma(M^2)$

A.1 Parametrization I

The principal part of Equation (II.1) is evaluated with $F(k^2)$ given by (II.8). Only $n=\frac{1}{2}$ is done here.

One makes the substitutions

$$k = \frac{mx}{(1-x^2)^{\frac{1}{2}}} \quad ; \quad dk = \frac{m\,dx}{(1-x^2)^{3/2}} \quad ;$$

$$m_0^2 = \frac{M^2}{4} - m^2 \quad ; \quad a^2 = \frac{s_0}{s_0 - 4m^2} \quad ; \quad b^2 = \frac{m_0^2}{m^2 + m_0^2} \quad .$$

Then one can write

$$\text{Re}\ \Sigma(M^2) = \frac{m^2a^2b^2}{\pi m_0^2}\, P\int_0^1 \frac{x^2\,dx}{(a^2 - x^2)(x^2 - b^2)} \quad . \tag{A.1}$$

When separated into partial fractions, the integrals can be found in tables(31). The final result is

$$\text{Re}\ \Sigma(M^2) = \frac{m^2a^2b^2}{2\pi m_0^2(a^2 - b^2)}\left(a\ \ln\frac{a+1}{a-1} + b\ \ln\frac{1-b}{1+b}\right) \tag{A.2}$$

A.2 Parametrizations II and III

For both Parametrizations II and III, the integral (II.1) has the form

$$\mathrm{Re}\,\Sigma(M^2) = \frac{P}{\pi}\int_0^\infty \frac{\alpha^2\,k^2\,dk}{\left\{(\alpha - k^2)^2 + \beta^2 k^2\right\}(k^2 + m^2)^{\frac{1}{2}}(k^2 - m_0^2)} , \tag{A.3}$$

where α can be positive or negative. With the substitutions

$$\omega = \alpha - \frac{\beta^2}{2} \quad ; \quad \omega_0 = \sqrt{\frac{\beta^2}{4} - \alpha} ,$$

the first factor in the denominator of the integrand can be written as

$$(k^2 - \omega - \omega_0)(k^2 - \omega + \omega_0) ,$$

and the integrand can be separated into partial fractions, and each of the resulting integrals has the same form as the integral in Appendix A.1. Then, putting

$$\gamma_1^2 = -\omega - \omega_0 \quad ; \quad \gamma_2^2 = -\omega + \omega_0$$

$$a_1^2 = \frac{\gamma_1^2}{\gamma_1^2 - m^2} \quad ; \quad a_2^2 = \frac{\gamma_2^2}{\gamma_2^2 - m^2} \quad ; \quad b^2 = \frac{m_0^2}{m^2 + m_0^2} ,$$

one obtains the relation

$$\mathrm{Re}\,\Sigma(M^2) = \frac{\alpha^2}{-2\omega_0\pi(\gamma_1^2 + m_0^2)}\, P\left(\int_0^1 \frac{a_1^2\,dx}{x^2 - a_1^2} - \int_0^1 \frac{b^2 dx}{x^2 - b^2}\right) \tag{A.4}$$

$+$ a similar expression with γ_2^2 and a_2^2 .

The final expression is then

$$\mathrm{Re}\,\Sigma(M^2) = \frac{\alpha^2}{4\omega_0\pi(\gamma_1^2 + m_0^2)(\gamma_2^2 + m_0^2)}\left(a_1(\gamma_2^2 + m_0^2)\ \ln\frac{1 + a_1}{1 - a_1} - a_2(\gamma_1^2 + m_0^2)\ \ln\frac{1 + a_2}{1 - a_2} - 2\omega_0 b\ \ln\frac{1 + b}{1 - b}\right) \tag{A.5}$$

One sees that this is extremely complicated and therefore it was found more convenient to use numerical methods to calculate Re $\Sigma(M^2)$. Notice that γ^2 and a^2 are, in general, complex, but Re $\Sigma(M^2)$ is real since Im γ_1^2 = -Im γ_2^2 and Im a_1 = -Im a_2 .

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APPENDIX B

Method of Handling the Singularity in the Numerical Calculation of Re $\Sigma(M^2)$

A principal part integration cannot be performed directly by numerical methods. Therefore one must find a method for computing

$$I = P \int_{a-\varepsilon}^{a+\varepsilon} \frac{f(x)}{x - a} dx \quad , \qquad (A.6)$$

with ε large enough to allow the integration outside the above limits to be done numerically. Making the substitution $\xi = x - a$ and expanding $f(x)$ in a Taylor series about a, the expression (A.6) becomes

$$I = P \int_{-\varepsilon}^{\varepsilon} \frac{f(a) + \xi f'(a) + \frac{\xi^2}{2} f''(a) + \cdots}{\xi} d\xi \quad . \qquad (A.7)$$

If ε is sufficiently small so that f and its derivatives do not change much in the range 2ε, then $I \simeq 2\varepsilon f'(a)$, or

$$I = f(a + \varepsilon) - f(a - \varepsilon).$$

In this problem, ε was chosen to be $0.05\ m^2$, and for the form factors used here, the approximation is excellent.

APPENDIX C

Calculation of $\delta(s)$

C.1 Parametrization I

F(s) is given by Equation (II.8). Evaluate Equation (II.6) using

$$g(s') = \frac{n}{s'-4m^2} \ln\left(1 + \frac{s'-4m^2}{s_0}\right).$$

Then, one has

$$(s-4m^2)^{\frac{1}{2}} f(s) = \frac{P}{\pi} \int_{4m^2}^{\infty} \frac{\ln\left(1 + \frac{s'-4m^2}{s_0}\right)}{(s'-4m^2)^{\frac{1}{2}}(s'-s)} ds'$$

$$+ \frac{P}{\pi} \int_{4m^2}^{\infty} \frac{(s'-4m^2)^{\frac{1}{2}} g(s') ds'}{s'} + C \quad .$$

The second integration is a constant which is cancelled by C. (Here, the boundary condition (II.7) is used.) Then, making the substitution $s' = u^2 + 4m^2$, one obtains

$$(s-4m^2)^{\frac{1}{2}} f(s) = 2n \frac{P}{\pi} \int_0^{\infty} \frac{\ln\left(\frac{u^2}{s_0} + 1\right)}{u^2 - s + 4m^2} du \quad ,$$

which can be found from tables(31). The final result is

$$\delta(s) = 2n \arctan\left(\frac{s - 4m^2}{s_0} \right)^{\frac{1}{2}} . \qquad (A.8)$$

C.2 Parametrization II

Making the substitution $s = u^2 + 4m^2$ and using the fact that

$$P \int_0^{\infty} \frac{du'}{u'^2 - u^2} = 0 ,$$

the integrand in Equation (II.6) can be separated into partial fractions and the right-hand side can be written

$$\text{R.H.S.} = \text{constant} - \frac{P}{\pi} \int_0^{\infty} \frac{\ln\left((u_r^2 - u'^2)^2 + 4u'^2\gamma^2\right) du'}{u'^2 - u^2} ,$$

where $u_r = (s_r - 4m^2)^{\frac{1}{2}} = 2k_r$

and $u = (s - 4m^2)^{\frac{1}{2}} = 2k$.

Then, changing the variables to $x = u/\gamma$, defining $\omega = \sqrt{x_r^2 - 1}$, and noting that the integral is symmetric in u' (or in x'), one has

$$\text{R.H.S.} = \frac{1}{2\pi\gamma} P \int_{-\infty}^{\infty} \frac{\ln\left((x' - \omega)^2 + 1\right) dx'}{x'^2 - x^2} \qquad (A.9)$$

plus the same expression with $\omega \to -\omega$. This expression can be evaluated by taking the real part of the appropriate contour integration. Now, the expression (A.9) can be separated into four contour integrations, each of which has a cut which must not be crossed by the contour. The four cuts start at $\omega+i$, $\omega-i$, $-\omega+i$, and $-\omega-i$, and extend to infinity without crossing the real axis. The contours for the integrations corresponding to the first and third cuts must be closed in the lower half-plane and the other two must be closed in the upper half-plane.

Performing the contour integrations, one finds

$$\text{R.H.S.} = \frac{i}{2\gamma x} \ln \frac{(x - \omega + i)(x + \omega + i)}{(x - \omega - i)(x + \omega - i)} .$$

Then, using the identities

$$\frac{1}{2i} \ln \frac{1 + iy}{1 - iy} = \arctan y ,$$

and $$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy} ,$$

substituting back in terms of the original variables and applying the boundary condition (II.7), one finally obtains

$$\delta(s) = \arctan \left(\frac{2\gamma\sqrt{s - 4m^2}}{s_r - s} \right) = \arctan \left(\frac{\gamma k}{k_r^2 - k^2} \right) . \qquad (A.10)$$

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C.3 Parametrization III

The form factor for Parametrization III is the same as that for Parametrization II with $\gamma/k_r^2 \rightarrow a$, and $k_r^2 \rightarrow 2/ar_0$. Then

$$\delta(k) = \arctan\left(\frac{ak}{1 + \tfrac{1}{2}ar_0 k}\right) \tag{A.11}$$

C.4 Parametrization IV

C.4.a $G(s) = \alpha$

With the substitution $s = u^2 + 4m^2$, Equation (II.6) can be written

$$\text{R.H.S.} = \frac{-2s\alpha}{\pi} \, P\int_0^\infty \frac{u'^2\,du'}{(u'^2 - s + 4m^2)(u'^2 + 4m^2)} \; .$$

This can be found in tables$^{(31)}$, and the result is

$$\delta(s) = \beta(s - 4m^2)^{\frac{1}{2}} \quad ,$$

where $\beta = 2m\alpha + C$, which cannot be determined from the boundary condition (II.7).

C.4.b $G(s) = \dfrac{\alpha}{(s - 4m^2)^{\frac{1}{2}}}$

The substitution $s = x + 4m^2$ and a separation of (II.6) into partial fractions leads to integrals which are found in tables$^{(31)}$. The result is

$$\delta(s) = \frac{\alpha}{\pi} (s - 4m^2)^{\frac{1}{2}} \ln\left(\frac{s - 4m^2}{4m^2}\right) . \qquad (A.12)$$

C.4.c $\quad G(s) = \dfrac{\alpha}{\sqrt{s}}$

The substitution $s = 4m^2/(1 - x^2)$ leads to the expression

$$(s - 4m^2)^{\frac{1}{2}} f(s) = \frac{-2\alpha P}{\pi} \int_0^1 \frac{x^2 \, dx}{x^2 - \left(1 - \frac{4m^2}{s}\right)} ,$$

which can be found in tables. Then, applying the boundary condition (II.7), one has

$$\delta(s) = \frac{\alpha}{\pi} \frac{(s - 4m^2)}{s^{\frac{1}{2}}} \ln\left(\frac{s^{\frac{1}{2}} + (s - 4m^2)^{\frac{1}{2}}}{s^{\frac{1}{2}} - (s - 4m^2)^{\frac{1}{2}}}\right) . \qquad (A.13)$$

APPENDIX D

Relation of γ to Full Width Γ

In the resonance parametrization one may write

$$F^2 \sim \frac{1}{(\omega_r - \omega)^2 + \dfrac{16k^2\gamma^2}{(\omega_r + \omega)^2}} ,$$

where ω_r is the mass of the resonance and $\omega = \sqrt{s}$ is the centre-of-mass energy of the two-particle system. Now define α and $\omega_{\frac{1}{2}}$ by

$$\omega_{\frac{1}{2}} \equiv \omega_r + \frac{\Gamma}{2} \equiv \omega_r(1 + \alpha) .$$

Then, to first order in α , $k_{\frac{1}{2}}^2 = k_r^2 + \dfrac{\alpha\omega_r^2}{2}$. Also, $(\omega_{\frac{1}{2}} + \omega_r)^2 = 4\omega_r^2(1 + \alpha)$ and $(\omega_{\frac{1}{2}} - \omega_r)^2 = \omega_r^2\alpha^2$. Now,

$$\tfrac{1}{2} F^2_{max} = \frac{1}{\dfrac{8k_r^2\gamma^2}{\omega_r^2}} ,$$

and, equating this to the value of F^2 at $\omega = \omega_{\frac{1}{2}}$, one has

$$\frac{8k_r^2\gamma^2}{\omega_r^2} = \omega_r^2\alpha^2 + \frac{4(k_r^2 + \dfrac{\alpha\omega_r^2}{2})\gamma^2}{\omega_r^2(1 + \alpha)} .$$

This can be written in the form

$$\Gamma^2 = \frac{4(\omega_r^2 - 4m^2)\gamma^2}{\omega_r^2}\left[1 - \alpha\left(\frac{\omega_r^2}{2k_r^2} - 1\right)\right] \quad . \tag{A.14}$$

For narrow resonances, the second term on the right-hand side may be neglected. The broad resonances have been seen to occur at higher energies, where $\omega_r^2 \sim 4k_r^2$, so that

$$\frac{\omega_r^2}{2k_r^2} - 1 \sim 1 \quad .$$

Notice that the first term on the right-hand side of (A.14) is approximately $4\gamma^2$ at high energies. Therefore, one can make the approximation $\alpha \simeq \gamma/\omega_r$. Then, retaining terms only up to first order in γ/ω_r, one finally has

$$\Gamma \simeq 2\gamma\left(\frac{s_r - 4m^2}{s_r}\right)^{\frac{1}{2}}\left(1 - \frac{\gamma}{2\sqrt{s_r}}\right) \quad . \tag{A.15}$$

APPENDIX E

Using the N/D method$^{(29)}$, one writes

$$\frac{N}{D} = \frac{1}{1 + \frac{1}{2}ar_0k - iak} ,$$

where
$$N = \frac{\lambda}{k^2 + \alpha^2} \qquad (A.16)$$

approximates the left-hand cut (i.e. the dynamical properties of the system). λ is the strength of the interaction and $1/\alpha$ represents the range of the interaction. D can now be easily determined from the above relations. The zeroes of D represent bound states of the system when they occur in the range $k^2 < 0$ on the first Riemann sheet of k^2 (For a discussion of this point, see Reference (12)). The zeroes of D occur at

$$k = i\alpha\left(\frac{\lambda - 2\alpha}{\lambda + 2\alpha}\right) ,$$

which can be written in terms of a, r_0 and α as

$$\text{Im } k = - \frac{1}{\alpha r_0 a} . \qquad (A.17)$$

Now, $\alpha > 0$ since $1/\alpha$ represents the range of the forces,

so that if a and r_0 have opposite signs, this form factor generates bound states. It was found that for $\alpha > 0$, only the case $a < 0$, $r_0 > 0$ could occur, and not $a > 0$, $r_0 < 0$. If $r_0 < 0$, $a > 0$, the form factor has the form of a resonance; therefore, a simple model for the forces like (A.16) can never generate a resonance. However, for more complicated models, possibly $a > 0$, $r_0 < 0$ could occur.

APPENDIX F

Calculation of the Integral (III.4)

Defining the quantities $\omega = P/2$ and $k' = k - P/2$, Equation (III.4) may be rewritten as

$$I = \frac{-i}{2(2\pi)^4} \int \frac{d^4k'}{\big((k'-\omega)^2 - m^2\big)\big((k'+\omega)^2 - m^2\big)} \quad .$$

If one now defines $\omega'^2 = k'^2 + m^2$, the poles of the left-hand term in the denominator are at

$$(k'^0 - \omega)^2 - \vec{k}'^2 - m^2 + i\varepsilon = 0$$

or

$$k'^0 = \omega \pm \omega' \mp i\varepsilon \quad ,$$

and for the right-hand term the poles are at $k' = -\omega \pm \omega' \mp i\varepsilon$. Then, after doing the k^0 contour integration by closing the contour in the lower half-plane, one is left

$$I(\omega) = \frac{1}{16\pi^2} \int_m^\infty \frac{k'd\omega'}{\omega' + \omega} \left(\frac{P}{\omega' - \omega} + i\pi\delta(\omega' - \omega) \right) \quad ,$$

where the part in brackets arises because the k' integrand is singular. This integral is logarithmically divergent so that a subtraction is made at the point $\omega = \omega_0$, $k = k_0$.

One makes the substitutions

$$x = \frac{k}{\omega} = \frac{\sqrt{\omega^2 - m^2}}{\omega}$$

and similar substitutions defining x' and x_0, and introduces a cutoff in k at Λ, or at $x = x(\Lambda)$. Then, after separating the integrand into partial fractions, one obtains

$$\mathrm{Re}\ I(\omega) \quad \frac{1}{16\pi^2}\left(\frac{x}{2}\ln\frac{1-x}{1+x} - \tfrac{1}{2}\ln\frac{x(\Lambda)-1}{2}\right.$$

$$\left. - \frac{x}{2}\ln\frac{1-x}{1+x} + \tfrac{1}{2}\ln\frac{x(\Lambda)-1}{2}\right),$$

where $x(\Lambda) \rightarrow 1$. The divergence is removed by the introduction of the subtraction term. Then, substituting back in terms of k and ω, the final result is

$$\mathrm{Re}\ I(\omega) = \frac{k}{\omega}\ln\frac{\omega - k}{m} - \frac{k_0}{\omega_0}\ln\frac{\omega_0 - k_0}{m} \ . \qquad (A.18)$$

The imaginary part of $I(\omega)$ can be calculated immediately because of the δ-function. The result is

$$\mathrm{Im}\ I(\omega) = \frac{1}{32\pi}\,\frac{k}{\omega} \ . \qquad (A.19)$$

This term can be neglected in Chapter III.

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